Journal of Sound and Vibration 327 (2009) 299–321

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

Boundary control of three-dimensional inextensible marine risers

K.D. Do *, J. Pan

School of Mechanical Engineering, The University of Western Australia, 35 Stirling Highway, Crawley, WA 6009, Australia

ARTICLE INFO

Article history:
Received 31 July 2008
Received in revised form
22 June 2009
Accepted 13 July 2009
Handling Editor: L.G. Tham
Available online 4 August 2009

ABSTRACT

This paper present a design of boundary controllers actuated by hydraulic actuators at the top end for global stabilization of a three-dimensional riser system. First, a set of partial and ordinary differential equations describing motion of the riser and hydraulic systems is developed. Second, several important properties of the riser system are derived. Based on these properties, we show that the conventional formula to calculate the riser effective tension is oversimplified and a new formula is provided. Next, boundary controllers are designed based on Lyapunov's direct method, the backstepping technique, the derived properties of the riser system dynamics, and Poincare's inequalities. Finally, the Galerkin approximation method is used to prove existence and uniqueness of the solutions of the closed loop control system.

1. Introduction

The need for production of oil and/or gas from the sea bed has made control of the dynamics of a marine riser, which is a structure connecting a oil and/or gas offshore platform with a well at the sea bed, a necessity for both ocean and control engineers. In general, the riser is subject to nonlinear deformation dependent hydrodynamic loads induced by waves, ocean currents, control forces exerted at the top, distributed/concentrated buoyancy from attached modules, its own weight, inertia forces and distributed/concentrated torsional couples. Before reviewing control techniques for the flexible marine risers, we here mention some early work on static analysis of the risers. In [1–3], the static models of both two- and three-dimensional risers are first presented based on the work in [4]. Then numerical simulations are carried out to analyze the effect of the system parameters on the riser equilibria. It should be also mentioned the recent work in [5], where the authors carry out static stability of a riser based on the variational method. Since the riser dynamics is essentially a distributed system and its motion is governed by a set of partial differential equations (PDE) in both time and space variables, modal control and boundary control approaches are often used to control the riser in the literature.

The modal control approach, see [6,7], involves with controlling a certain number of modes of a distributed system. Basically, a distributed system is discretized to obtain a lumped-parameter system described in terms of modal coordinates. The advantage of this approach is that many available control design techniques, see [8,9], can be applied to design various controllers for the resulting lumped-parameter system. However, there are two main disadvantages of the modal control approach. The first drawback is difficulty in computing infinite dimensional gain matrices. This difficulty can be avoided by using the independent modal-space control method, but this method requires a distributed control force, which is impractical to implement. In practice, a truncated model consisting of a limited number of modes is usually used. However, the truncated model can be of a very large dimension to describe the behavior of a distributed system satisfactorily, i.e. it is impractical to control all modes. Therefore, the second disadvantage of the modal approach is

* Corresponding author.
E-mail addresses: duc@mech.uwa.edu.au (K.D. Do), pan@mech.uwa.edu.au (J. Pan).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2009.07.009

Nomenclature

$a \cdot b$	dot product of vectors a and b
$a \times b$	cross product of vectors a and b
A_{iH}, $i=1,2,3$	ram area of the cylinder of the i hydraulic system
A_r	cross section area of the riser
b_{iH}, $i=1,2,3$	combined coefficient of the modeled damping and viscous friction
B	bending rigidity of the riser
C_{iHD}, $i=1,2,3$	discharge coefficient of the i hydraulic system
C_{iHT}, $i=1,2,3$	coefficient of the total internal leakage of the cylinder of the i hydraulic system
C_{LD}	linear drag coefficient
C_{ND}	nonlinear drag coefficient
E	Young's modulus of the riser
F	internal force vector
g_i, $i=1,2$	initial displacement and velocity vectors
G	torsional rigidity of the riser
H	initial torsional moment around the $\hat{t}$ axis
I_{iH}, $i=1,2,3$	current input to the i hydraulic system
I_r	second moment of the riser cross section area
k_{iHv}, $i=1,2,3$	servovalve gain of the i hydraulic system
K_E	kinetic energy
L_A	modified Lagrangian
m	external distributed moment vector
m_{iH}, $i=1,2,3$	mass of the piston of the i hydraulic system
m_o	oscillating mass of the riser per unit length
M	internal moment vector
$M_{\hat{b}}$	component of M along the $\hat{b}$ direction
$M_{\hat{n}}$	component of M along the $\hat{n}$ direction
$M_{\hat{t}}$	component of M along the $\hat{t}$ direction
P_E	potential energy
P_{i1}, $i=1,2,3$	pressure in upper compartment of the cylinder of the i hydraulic system
P_{i2}, $i=1,2,3$	pressure in lower compartment of the cylinder of the i hydraulic system
P_{iHS}, $i=1,2,3$	supply pressure of the i hydraulic system
q	external distributed force vector
Q_{iH}, $i=1,2,3$	load flow of the i hydraulic system
s	arc length of the riser center line
V_{iH}, $i=1,2,3$	total volume of the cylinder and hoses of the i hydraulic system
V_n	relative flow velocity normal to the riser
w	displacement vector of a riser center line point
w_{re}	effective riser weight per unit length
W_{iH}, $i=1,2,3$	spool area of the i hydraulic system
x_{iH}, $i=1,2,3$	position of the piston of i hydraulic system
β_{iHe}, $i=1,2,3$	effective modulus of the oil in the i hydraulic system
δW_c	variation of the virtual work
$\hat{\Delta}$	estimate of Δ
κ	curvature of the riser center line
λ_c	continuous Lagrangian multiplier
μ	shear modulus of the riser
ρ_{iH}, $i=1,2,3$	density of the oil inside the i hydraulic system
ρ_r	density of the riser
τ_{iHv}, $i=1,2,3$	time constant of the i hydraulic system
$\hat{b}$	unit vector in binormal direction
$\hat{n}$	unit vector in principal direction
$\hat{t}$	unit vector in tangent direction

restricted to control of a few critical modes. The other modes, which are not controlled, could be unstable. This can be understood as follows [10]. A truncation of a distributed system divides the system into three groups of modes: modeled and controlled, modeled and uncontrolled (residual), and un-modeled. The control design considers only the modeled modes. The output of these modeled modes is provided by observers from the actual distributed system, and is then fed to the control design. The use of these observers and truncated models of distributed system results in a spill-over phenomenon. This means that the control action from actuators affects not only the controlled modes but also influences the residual and un-modeled modes, which can be unstable.

In the boundary control approach, the original PDE model is considered and the boundary control inputs are implemented at the boundaries to control all the modes. Therefore, the boundary control approach is much more practical than the modal control approach in the sense that it excludes the effect of both observation and control spill-over phenomenon. In addition, no distributed actuators and sensors are required. The main tools used to design boundary controllers for a distributed system are functional analysis and semi-group theory, see [11,12], and the Lyapunov direct method, see [13,14]. The Lyapunov direct method is widely used since the control Lyapunov functions/functionals can be mimicked by those developed for discrete systems [13]. Using the Lyapunov direct method, various boundary controllers have been proposed for flexible beam-like systems, see [15–17] for boundary controllers to reduce transverse vibration of an axially moving string, [18–20] for boundary controllers stabilizing transverse motion of a beam. It is noted that in all the above boundary control designs, except for the one in [20], disturbance distributed forces including the structures' own weight are ignored, and no proof of existence and uniqueness of the solutions of closed loop systems was given. Recently, in [21–23] the authors proposed an elegant method, which was developed for stabilizing an unstable heat equation in [24], to design boundary controllers for strings and beams with simple dynamics. The fundamental idea is to find a coordinate change to transform the string or beam system to a target system, which can be stabilized by a boundary controller. However, the method in [21–23] is hard to apply to the riser system addressed in this paper due to difficulties in solving a partial differential equation to find a proper kernel.

In the above references, the beams or strings were assumed to deform in only one plane, and only transverse motion was considered and controlled in the above control designs. Mathematical work in [25] shows that even slight space curvature introduces significant changes in the beam natural frequencies and especially on mode shapes, i.e. the coupling of the out-of-plane wave types, and extensional and flexural waves exhibits in the flexible beams. The coupling between these wave types due to the curved shape of the riser, boundary constraints and external forces made the energy exchange from one wave type to other possible. Therefore, the control problem of a flexible marine riser that deforms in three-dimensional space is necessary.

In this paper, we consider a control problem of global stabilization for a three-dimensional nonlinear inextensible flexible marine riser system. The riser is controlled by hydraulic systems installed at the top end of the riser. This paper is not a straightforward extension of our work in [20] where the riser was restricted to move in one vertical plane, and only transverse motion was considered and controlled. In three-dimensional space, there are strong couplings between motions of a flexible marine riser along the *X*-, *Y*- and *Z*-axis, see Section 2.1. These couplings cause more difficulties to control a flexible marine riser in three-dimensional space than the one studied in [20]. As such, we propose to solve the control problem under consideration in several stages. First, a set of partial and ordinary differential equations and boundary conditions describing motion of the riser and hydraulic systems are developed based on balancing internal and external forces/moments, and the Hamilton principle. Second, various important properties of the equations of motion of the riser system are derived, see Lemma 1 of this paper. As a by-product of this derivation, we show that the conventional formula to calculate the riser effective tension is oversimplified and a new formula is provided, see Remark 4. The derived properties of the riser system dynamics and Poincare's inequalities are extensively used in bounding the derivatives of the Lyapunov function candidates, which are crucial for the success of the boundary controller design. Third, we use Lyapunov's direct method (where a nontrivial Lyapunov function candidate is proposed, see (35)), the backstepping technique, and Poincare's inequalities to design boundary controllers to stabilize the riser at its equilibrium position. The proposed controllers guarantee that when there are no environmental disturbances, the riser is globally exponentially stabilized at its equilibrium position, and that when there are environmental disturbances, the riser is stabilized in the neighborhood of its equilibrium position. Finally, the Galerkin approximation method is used to prove existence and uniqueness of the solutions of the closed loop control system.

2. Mathematical model and control objective

2.1. Mathematical model

In this section, we develop equations of motion of the riser and of the hydraulic systems. These equations will be used for the boundary control design in the next section. In developing the equations of motion of the riser, we make the following assumption:

Assumption 1. (1) The riser can be modeled as a beam rather than a shell since the diameter-to-length of the riser is small, i.e. we consider the riser as a slender structure.

(2) Plane sections remain plane after deformation, i.e. warping is neglected.
(3) The riser is locally stiff, i.e. cross sections do not deform and Poisson effect is neglected.
(4) The riser material is homogeneous, isotropic and linearly elastic, i.e. it obeys Hooke's law.
(5) The riser is initially straight and vertical.
(6) Torsional and distributed moments induced by environmental disturbances are neglected.
(7) The riser is inextensible.

Remark 1. Items (1)–(4) mean that the riser will be modeled as a Bernoulli-type of beam and not a Timoshenko-type, and that the extension of the riser axis small. Bernoulli–Euler models are satisfactory for modeling low frequency vibrations of beams. Item (5) generally holds in practice, and is made to simplify the development of the mathematical model and boundary controller. This item can be readily removed. Item (6) implies that we consider fluid/gas transportation risers rather than drilling risers, and that moment induced by the asymmetry of the relative flow due to vortex shedding is ignored.

2.1.1. Riser coordinate system

The riser system considered in this paper is presented in Fig. 1. The boundary forces exerted at the top of the riser along the *x*-, *y*- and *z*-axes are provided by three independent hydraulic systems installed on the ship/rig along the *x*-, *y*- and *z*-axes, respectively, see Figs. 1(a) and 1(b). The riser coordinates are presented in Fig. 1(a). In this figure, we have two coordinate systems. The earth-fixed system is (*OXYZ*), where *O* is the bottom ball-joint of the riser, and the *OZ* axis is along the initial riser. Let $r^0(s_0, t_0) = [x_0, y_0, z_0]$ be the position vector of the point P_0 of the initial riser centerline at the time t_0

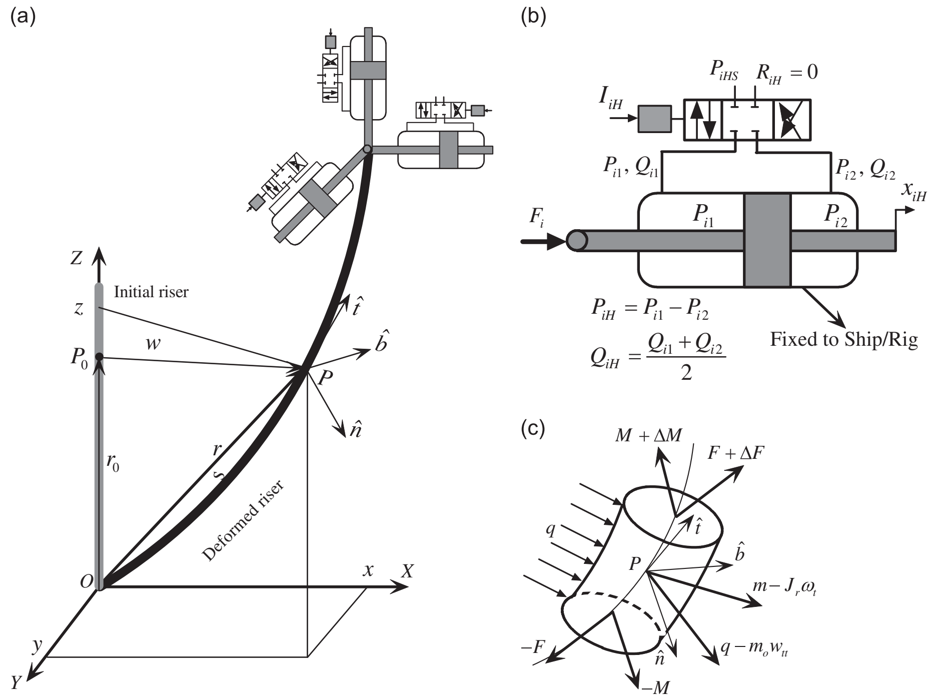

Fig. 1. General riser coordinate system, hydraulic system, and forces and moments acting on a riser element. (a) General riser coordinate system; (b) hydraulic system; (c) forces and moments on a riser element ds.

and the arc length s_0 from the point O. Hence at the time $t > t_0$, the point P_0 moves to the point P of the deformed riser centerline. The position of the point P is denoted by $r(s,t) = [x(s,t), y(s,t), z(s,t)]$ at the arc length s from the point O. Moreover, let $w(s,t) = [w_x(s,t), w_y(s,t), w_z(s,t)]^{\mathrm{T}}$ be the vector from the point P_0 to the point P. Then we have

$$r = r^0 + w \tag{1}$$

where from now onward whenever it is not confusing, we drop the arguments (t,s) and (t_0,s_0) of r, w and r^0, respectively for clarity. The body-fixed system is $(\hat{t}, \hat{n}, \hat{b})$, whose axes are the tangent, principal normal and binormal unit vectors. These vectors can be expressed in terms of the fixed system as

$$\hat{t} = r_s, \quad \hat{n} = \hat{t}_s/\kappa, \quad \hat{b} = \hat{t} \times \hat{n} \tag{2}$$

where the subscript s denotes the partial derivative with respect to the arc-length s, and κ is curvature of the riser center line at s depicting the rate of change of the orientation of the normal plane $(\hat{n}, \hat{b})$ defined by $\kappa = \|r_{ss}\|$. The above definition of the body-fixed coordinate system means that $(\hat{t}, \hat{n}, \hat{b})$ form a right handed orthonormal triad.

2.1.2. Equations of motion of the riser

Now from Fig. 1(c), balancing the forces and moments on a component ds of the deformed riser results in

$$m_o w_{tt} = F_s + q$$

$$J\omega_t = M_s + \hat{t} \times F + m \tag{3}$$

where from now onward, we use the subscript t to denote the partial derivative with respect to the time t, $m_o = \rho_r A_r$ is the oscillating mass of the riser per unit length with A_r being the riser cross section area, and ρ_r being the density of the riser, $J = \rho_r I_r$ with I_r being the second moment of the riser cross section area about the $\hat{b}$ axis, F and M are internal force and moment vectors, q and m are the external distributed force and moment vectors, and $\omega_t = \hat{n} \times \hat{n}_{tt} + \hat{b} \times \hat{b}_{tt}$ is the angular acceleration of a point on the centerline. The distributed moment vector m is induced by the asymmetry of the relative flow due to vortex shedding. Let $(M_{\hat{t}}, M_{\hat{n}}, M_{\hat{b}})$ be the components of M along the $\hat{t}, \hat{n}, \hat{b}$ axes of the body-fixed system, respectively. We then can write M as

$$M = M_{\hat{t}}\hat{t} + M_{\hat{n}}\hat{n} + M_{\hat{b}}\hat{b} \tag{4}$$

Since the riser is assumed to be straight at the initial time t_0, we have the following constitutive relations, see [4,26]:

$$M_{\hat{b}} = B\kappa, \quad M_{\hat{n}} = 0, \quad M_{\hat{t}} = G\tau + H \tag{5}$$

where $B = EI_r$ is the bending rigidity of the riser with E being Young's modulus; H is the initial torsional moment around the $\hat{t}$ axis; $G = 2\mu I_r$ is the torsional rigidity of the riser with μ being the shear modulus.

Since we neglect the torsional moment $G\tau + H$, distributed moment m and rotary inertia ρJ, the equations of motion of the riser given in (3) are simplified to

$$m_o w_{tt} = F_s + q$$

$$r_s \times (Bw_{sss} + F) = 0 \tag{6}$$

where we have used $M = M_{\hat{b}}\hat{b} = Br_s \times r_{ss}$ (see (2) and (4)), and the fact that $r_{sss} = w_{sss}$ due to the initial straight condition of the riser.

Remark 2. In [27], a local coordinate system (a_1, a_2, a_3) where a_3 coincides with $\hat{t}$, different from the local coordinate $(\hat{t}, \hat{n}, \hat{b})$ in this paper is used. Using the local coordinate (a_1, a_2, a_3) results in complexities in calculating the curvatures of the riser in the (a_1, a_3) and (a_2, a_3) planes. Indeed, one can rotate the coordinate system (a_1, a_2, a_3) round the $\hat{t}$ axis angle to have the coordinate system $(\hat{t}, \hat{n}, \hat{b})$. In [26], the constitutive equation for the moment in the normal direction, $M_{\hat{n}}$, is misgiven, since $M_{\hat{n}}$ is always zero for the riser under consideration.

Environmental disturbance vector q: The external disturbance vector q per unit length consists of fluid drag force, any concentrated forces exerted on the riser by attached cables and/or buoys modeled by Dirac functions, and effective riser weight defined as the weight of the riser plus contents in water. It is noted that the effective rather than the actual riser weight is used because the effective tension is used instead of the actual tension. In this paper, we do not consider cables or buoys attached to the riser. The fluid drag force is found by the use of a generalization of Morison's formula to account for cylinders, which are not oriented normal to the relative flow [28]. Taking the effective riser weight into account, we have

$$q(s,t,w_t,r_s) = \hat{t} \times (W_{re} \times \hat{t}) + \tfrac{1}{2}\rho_w C_{\mathrm{LD}} D_H V_n + \tfrac{1}{2}\rho_w C_{\mathrm{ND}} D_H \|V_n\| V_n \tag{7}$$

where C_{LD} and C_{ND} are the linear and nonlinear drag coefficients, respectively; D_H is the local riser hydrodynamic diameter; $W_{re} = -[0\ \ 0\ \ w_{re}]^{\mathrm{T}}$ with w_{re} is the effective riser weight per unit length; V_n is the component of the relative flow velocity normal to the riser centerline. Letting V be the (bounded) liquid flow velocity due to waves and currents. Then taking the riser motion into account, the relative flow velocity normal to the riser centerline, V_n, is given by

$$V_n = \hat{t} \times ((V - w_t) \times \hat{t}) = (I_{3\times 3} - r_s r_s^{\mathrm{T}})(V - w_t) \tag{8}$$

where $I_{3\times 3}$ is the three-dimensional identity matrix. Substituting (8) into (7) results in the equation for external disturbance vector q as follows:

$$\begin{aligned} q(s,t,w_t,r_s) = {} & (I_{3\times 3} - r_s r_s^{\mathrm{T}})W_{re} + \tfrac{1}{2}\rho_w C_{\mathrm{LD}} D_H (I_{3\times 3} - r_s r_s^{\mathrm{T}})(V - w_t) \\ & + \tfrac{1}{2}\rho_w C_{\mathrm{ND}} D_H \|(I_{3\times 3} - r_s r_s^{\mathrm{T}})(V - w_t)\|(I_{3\times 3} - r_s r_s^{\mathrm{T}})(V - w_t) \end{aligned} \tag{9}$$

Initial and boundary conditions: The initial conditions of the riser consist of the initial position and velocity functions. They are

$$w(s,t_0) = g_1(s), \quad w_t(s,t_0) = g_2(s), \quad \forall s \in (0,L) \tag{10}$$

where $g_1(s)$ and $g_2(s)$ are sufficiently smooth and bounded function vectors of s, and compatible with the boundary conditions. We first provide the kinetic and potential energies, modified Lagrangian, and variation of the virtual work done by nonconservative force q and by the virtual momentum transport at the boundary, then use the extended Hamilton principle to derive the boundary conditions.

The kinetic energy K_E of the riser and the pistons of the hydraulic systems, and the potential energy P_E of the riser with a length of L are

$$K_E = \frac{1}{2}\int_0^L m_o w_t \cdot w_t \, \mathrm{d}s + \frac{1}{2} w_t(L,t) M_H w_t(L,t)$$

$$P_E = \frac{1}{2}\int_0^L B w_{ss} \cdot w_{ss} \, \mathrm{d}s \tag{11}$$

where $M_H = \mathrm{diag}(m_{1H}, m_{2H}, m_{3H})$ with m_{1H}, m_{2H} and m_{3H} being the mass of the piston of the hydraulic system that provides the boundary force at the top end of the riser along the x-, y- and z-axis, respectively; $\mathrm{diag}(m_{1H}, m_{2H}, m_{3H})$ denotes the diagonal matrix with the diagonal elements being m_{1H}, m_{2H} and m_{3H}. Since the riser response must satisfy the kinetic constraint of the unit tangent vector $\hat{t}$, i.e. $r_s \cdot r_s = 1$ in terms of deformation applying along the riser, the modified Lagrangian L_A of the riser is given as follows:

$$L_A = K_E - P_E + \frac{\lambda_c}{2}\int_0^L (r_s \cdot r_s - 1) \, \mathrm{d}s \tag{12}$$

where λ_c is the continuous Lagrangian multiplier. To derive the boundary conditions, we now use the following extended Hamilton principle:

$$\int_{t_1}^{t_2} (\delta L_A + \delta W_c + \delta W_b)\,\mathrm{d}t = 0$$

$$\delta w(s,t_1) = \delta w(s,t_2) = 0 \tag{13}$$

where t_1 and t_2 denote time, δW_c is variation of the virtual work done by nonconservative force, and δW_b is variation of the virtual work done by the virtual momentum transport at the boundary. The variation of the virtual work δW_c done by nonconservative force $q(s,t,w_t,r_s)$ is given by

$$\delta W_c = \int_0^L q(s,t,w_t,r_s)\delta w(z,t)\,\mathrm{d}z \tag{14}$$

The variation of the virtual work δW_b done by the virtual momentum transport at the boundary is given by

$$\delta W_b = (A_H P_H - \varDelta(t, w_t(L,t)) - B_H w_t(L,t))\delta w(L,t) \tag{15}$$

where

$$P_H = [P_{11} - P_{12}, P_{21} - P_{22}, P_{31} - P_{32}]^{\mathrm{T}}$$

$$A_H = \mathrm{diag}(A_{1H}, A_{2H}, A_{3H})$$

$$B_H = \mathrm{diag}(b_{1H}, b_{2H}, b_{3H})$$

$$\varDelta(t, w_t(L,t)) = [\varDelta_1(t, w_t(L,t)), \varDelta_2(t, w_t(L,t)), \varDelta_3(t, w_t(L,t))]^{\mathrm{T}} \tag{16}$$

In (16), P_{i1} with $i = 1,2,3$ and P_{i2} are the pressures in the upper and lower compartments of the cylinder i, see Fig. 1(b), A_{iH} is the ram area of the cylinder i, b_{iH} represents the combined coefficient of the modeled damping and viscous friction forces on the cylinder rod i, and $\varDelta_i(t, w_t(L,t))$ is the un-modeled force acting on the cylinder i of the hydraulic system i. This un-modeled force can include un-modeled friction between the cylinder and the piston of the hydraulic system i, and the external disturbance from the cylinder of the hydraulic system i acting on the piston i of the hydraulic system i. It is noted that all the cylinders of the hydraulic systems can be either fixed to the vessel/rig or an active heave compensation system fixed to the vessel/rig, see [29] for more details. The vessel/rig is stabilized at its desired location by a separating dynamic positioning system. Since many dynamic positioning systems are available in the literature, see [30], we do not include the dynamics of the vessel/rig in this paper. However, we take effects of motion of the vessel/rig around its equilibrium point on the riser through the disturbance $\varDelta_i(t, w_t(L,t))$. Substituting (15), (14) and (11) into (13) and using the boundary specifications of the riser under consideration result in

$$m_o w_{tt} = F_s + q, \quad s \in (0,L)$$

$$r_s \times (B w_{sss} + F) = 0, \quad s \in (0,L)$$

$$M_H w_{tt}(L,t) = -B_H w_t(L,t) - F(L,t) + A_H P_H - \varDelta(t, w_t(L,t)),$$

$$w(0,t) = 0, \quad w_{ss}(0,t) = 0, \quad w_{ss}(L,t) = 0 \tag{17}$$

where we have taken $\lambda_c = F \cdot r_s - B\kappa^2$ motivated by (6).

Remark 3. The riser dynamics (17) is one-dimensional (with respect to the spatial variable s). This means that a point on the riser cross section, other than the point on the centerline, cannot be traced after deformation takes place. In this paper, we consider the deformation of the riser centerline, which is, in general, a three-dimensional space curve.

2.1.3. Equations of motion of the hydraulic systems

The second equation in (17) represents the dynamics of the pistons of the hydraulic systems with

$$w(L,t) = x_H$$

$$w_t(L,t) = \dot{x}_H \tag{18}$$

where $x_H = [x_{1H}, x_{2H}, x_{3H}]^{\mathrm{T}}$ is the position vector of the pistons of the hydraulic system, see Fig. 1(b). Neglecting the leakage flows in the cylinder and the servovalve, the actuator or the cylinder dynamics is written as [31]

$$\bar{V}_H \dot{P}_H = -A_H \dot{x}_H - C_{HT} P_H + Q_H \tag{19}$$

where

$$\bar{V}_H = \mathrm{diag}\left(\frac{V_{1H}}{4\beta_{1He}}, \frac{V_{2H}}{4\beta_{2He}}, \frac{V_{3H}}{4\beta_{3He}}\right)$$

$$C_{HT} = \mathrm{diag}(C_{1HT}, C_{2HT}, C_{3HT})$$

$$Q_H = [Q_{1H}, Q_{2H}, Q_{3H}]^{\mathrm{T}} \tag{20}$$

In (20), $V_{iH},\ i = 1,2,3$ is the total volume of the cylinder i and the hoses between the cylinder i and the servovalve i, β_{iHe} is the effective bulk modulus, C_{iHT} is the coefficient of the total internal leakage of the cylinder i due to pressure, Q_{iH} is the load flow of the hydraulic system i. The load flow vector Q_H is related to the spool displacement vector of the servovalve, $x_{H\nu}$, by [31]

$$Q_H = \Psi x_{H\nu} \tag{21}$$

where $\Psi = \mathrm{diag}(\Psi_1, \Psi_2, \Psi_3)$ with

$$\Psi_i = C_{iHD} W_{iH} \sqrt{\frac{P_{iHS} - \tanh(x_{iH\nu}/\sigma_i) P_{iH}}{\rho_{iH}}}$$

Here, $C_{iHD},\ i = 1,2,3$ is the discharge coefficient, W_{iH} is the spool valve area gradient, P_{iHS} is the supply pressure of the fluid, σ_i is a small positive constant, and ρ_{iH} is density of the oil of the hydraulic system i. It is noted that since the supply pressure P_{iHS} is always higher than the load pressure P_{iH}, i.e. there exists a strictly positive constant ε such that $P_{iHS} - \tanh(x_{iH\nu}/\sigma_i)P_{iH} \geq \varepsilon$. Hence, Eq. (21) is well-defined for all $x_{H\nu} \in \mathbb{R}^3$. Moreover, the function $\tanh(x_{iH\nu}/\sigma_i)$ has been used to replace the signum function $\mathrm{sgn}(x_{iH\nu})$ originated in [31]. It is noted that the use of the function $\tanh(x_{iH\nu}/\sigma_i)$ not only makes the function Ψ_i differentiable with respect to $x_{iH\nu}$ but also represents the actual dynamics of the spool dynamics. This is because there is always certain round-off of sharp edges in manufacturing the servovalve, i.e. the flow in the servovalve does not change its direction immediately. The servovalve dynamics can be described as

$$T_{H\nu}\dot{x}_{H\nu} = -x_{H\nu} + K_{H\nu} I_H \tag{22}$$

where

$$T_{H\nu} = \mathrm{diag}(\tau_{1H\nu}, \tau_{2H\nu}, \tau_{3H\nu})$$

$$K_{H\nu} = \mathrm{diag}(k_{1H\nu}, k_{2H\nu}, k_{3H\nu})$$

$$I_H = \mathrm{diag}(I_{1H}, I_{2H}, I_{3H}) \tag{23}$$

with $\tau_{iH\nu},\ i = 1,2,3$ and $k_{iH\nu}$ are the time constant and gain of the servovalve i, respectively, I_{iH} is the current input to the hydraulic system i. We now write the equations of motion of the riser and the hydraulic systems consisting of (17), (20), (21) and (22) in a standard form for control design in the next section as follows:

$$
\begin{aligned}
& m_o w_{tt} = F_s + q, \quad s \in (0,L) \\
& r_s \times (Bw_{sss} + F) = 0, \quad s \in (0,L) \\
& \dot{x}_1 = x_2 \\
& \dot{x}_2 = M_H^{-1}(-B_H x_2 - F(L,t) + A_H x_3 - \Delta(t, x_2)) \\
& \dot{x}_3 = \bar{V}_H^{-1}(-A_H x_2 - C_{HT} x_3 + \Psi x_4) \\
& \dot{x}_4 = T_{H\nu}^{-1}(-x_4 + K_{H\nu} I_H) \\
& w(0,t) = 0, \quad w_{ss}(0,t) = 0, \quad w_{ss}(L,t) = 0
\end{aligned} \tag{24}
$$

where we have defined

$$x_1 = w(L,t), \quad x_2 = w_t(L,t), \quad x_3 = P_H, \quad x_4 = x_{H\nu} \tag{25}$$

Moreover, we have the following results, which will be used extensively in the control design in the next section.

Lemma 1. *For the riser dynamics* (*the first two equations and the last equation of* (24)) *and under the inextensible condition of the riser, the following equations hold*:

$$w_s(s,t) \cdot r_s(s,t) = \tfrac{1}{2} w_s(s,t) \cdot w_s(s,t), \quad \forall (s,t) \in ([0,L], \mathbb{R}^+) \tag{26}$$

$$w_s(s,t) \cdot w_s(s,t) \leq 2, \quad \forall (s,t) \in ([0,L], \mathbb{R}^+) \tag{27}$$

$$F(s,t)\cdot w_{ss}(s,t) = -Bw_{sss}(s,t)\cdot w_{ss}(s,t), \quad \forall (s,t)\in([0,L],\mathbb{R}^+) \tag{28}$$

$$\begin{aligned} F(s,t)\cdot w_s(s,t) = & -Bw_{sss}(s,t)\cdot w_s(s,t) + F(s,t)\cdot r_s(s,t) w_s(s,t)\cdot r_s(s,t) \\ & + Bw_{sss}(s,t)\cdot r_s(s,t) r_s(s,t)\cdot w_s(s,t), \quad \forall (s,t)\in([0,L],\mathbb{R}^+) \end{aligned} \tag{29}$$

$$\begin{aligned} F(s,t)\cdot r_s(s,t) = F(L,t)\cdot r_s(L,t) - \frac{B}{2} w_{ss}(s,t)\cdot w_{ss}(s,t) \\ + \int_s^L q(\sigma,t,w_t(\sigma,t),r_\sigma(\sigma,t))\cdot r_\sigma(\sigma,t)\,\mathrm{d}\sigma, \quad \forall (s,t)\in((0,L),\mathbb{R}^+) \end{aligned} \tag{30}$$

$$\begin{aligned} F_s(s,t)\cdot w_{tt}(s,t) = & -Bw_{ssss}(s,t)\cdot r_s(s,t) + Bw_{sss}(s,t)\cdot r_s(s,t) w_{tt}(s,t)\cdot w_{ss}(s,t) \\ & + F(s,t)\cdot r_s(s,t) w_{tt}(s,t)\cdot w_{ss}(s,t), \quad \forall (s,t)\in([0,L],\mathbb{R}^+) \end{aligned} \tag{31}$$

$$r_s(L,t).w(L,t) = \int_0^L w_{ss}(s,t)\cdot w(s,t)\,\mathrm{d}s + \frac{1}{2}\int_0^L w_s(s,t)\cdot w_s(s,t), \quad \forall (s,t)\in((0,L),\mathbb{R}^+)\,\mathrm{d}s \tag{32}$$

$$(F(s,t) + Bw_{sss}(s,t))\cdot w_{st}(s,t) = 0 \tag{33}$$

Proof. See Appendix A.

Remark 4. Since $F(s,t)\cdot r_s(s,t)$ is the actual tension at the point P, see Fig. 1(a), and at the time t, Eq. (30) is a formula that can be used to calculate the actual tension of the riser at any point along the riser center line and at any time t. This equation also indicates that the actual tension in the riser depends on the curvature of the riser center line due to the term $-(B/2)w_{ss}(s,t)\cdot w_{ss}(s,t)$. In existing literature [26], the formula for calculating the riser actual tension is oversimplified in the sense that the curvature of the riser is not included. Noticing that the magnitude of the term $-(B/2)w_{ss}(s,t)\cdot w_{ss}(s,t)$ is not necessarily small since the bending stiffness B can be large despite of small curvature $\|w_{ss}(s,t)\|$.

2.2. Control objectives

Under Assumption 1, design the control I_H for the riser-hydraulic system (24) to stabilize the riser at its vertical position in the sense that all the states of the riser-hydraulic system are bounded and that:

(1) when the external disturbance vector q is ignored, all the terms $\|w(s,t)\|$, $\int_0^L w_s(s,t)\cdot w_s(s,t)\,\mathrm{d}s$, $\int_0^L w_t(s,t)\cdot w_t(s,t)\,\mathrm{d}s$ and $\int_0^L w_{ss}(s,t)\cdot w_{ss}(s,t)\,\mathrm{d}s$ exponentially converge to zero for all $s\in[0,L]$ and $t\geq t_0$,
(2) when the external disturbance vector q is present, all the terms $\|w(s,t)\|$, $\int_0^L w_s(s,t)\cdot w_s(s,t)\,\mathrm{d}s$, $\int_0^L w_t(s,t)\cdot w_t(s,t)\,\mathrm{d}s$ and $\int_0^L w_{ss}(s,t)\cdot w_{ss}(s,t)\,\mathrm{d}s$ exponentially converge to some small positive constants for all $s\in[0,L]$ and $t\geq t_0$.

It is seen that the control objective imposes on both the displacement and integration of square of the slope, velocity, and curvature of the riser along the riser length.

3. Boundary control design

A close look at the entire system (24) shows that the system is of a strict-feedback form [9]. Therefore, we will use the backstepping technique [9] to design the control input I_H to achieve the control objective stated in the previous section. The control design consists of three steps as follows.

3.1. Step 1

At the this step, we consider the hydraulic force $A_H P_H$, i.e. $A_H x_3$, as a control to design a boundary control law (i.e. a control law only uses $w(L,t)$ and its spatial and time derivatives) such that it stabilizes the riser at a small neighborhood of its vertical position. Ideally, we want to stabilize the riser at its vertical position but this is impossible due to the distributed external disturbances q induced by waves, wind and ocean currents. As such, we define

$$x_{3e} = A_H x_3 - \alpha_1 \tag{34}$$

where α_1 is a virtual control of $A_H x_3$. To design the virtual boundary control α_1, we use Lyapunov's direct method. Consider the following Lyapunov function candidate:

$$\begin{aligned} W_1 = & \frac{m_o}{2}\int_0^L w_t\cdot w_t\,\mathrm{d}s + \frac{B}{2}\int_0^L w_{ss}\cdot w_{ss}\,\mathrm{d}s + \frac{\lambda}{2}\int_0^L w_s\cdot w_s\,\mathrm{d}s + \gamma\int_0^L s w_t\cdot w_s\,\mathrm{d}s - \frac{\gamma}{2}\int_0^L w_t\cdot w\,\mathrm{d}s \\ & + \frac{1}{2}\left[w_t(L,t) + \frac{\gamma L}{m_o} w_s(L,t) - \frac{\gamma}{2m_o} w(L,t)\right]^{\mathrm{T}} M_H \left[w_t(L,t) + \frac{\gamma L}{m_o} w_s(L,t) - \frac{\gamma}{2m_o} w(L,t)\right] \end{aligned} \tag{35}$$

where λ and γ are positive constants to be specified later. Since for all $t \geq t_0$, we have

$$\left|\int_0^L s w_t \cdot w_s \, \mathrm{d}s - \frac{1}{2}\int_0^L w_t \cdot w \, \mathrm{d}s\right| \leq \frac{L+1}{2}\int_0^L w_t \cdot w_t \, \mathrm{d}s + \frac{L+L^2}{2}\int_0^L w_s \cdot w_s \, \mathrm{d}s \tag{36}$$

where we have used completion of squares and Lemma 2, see Appendix B, to obtain $\int_0^L w \cdot w \, \mathrm{d}s \leq 4L^2 \int_0^L w_s \cdot w_s \, \mathrm{d}s$ since $w(0,t) = 0$. Therefore, the function W_1 satisfies

$$\begin{aligned} W_1 \geq\ & \frac{m_o - \gamma(L+1)}{2}\int_0^L w_t \cdot w_t \, \mathrm{d}s + \frac{B}{2}\int_0^L w_{ss} \cdot w_{ss} \, \mathrm{d}s + \frac{\lambda - \gamma(L+L^2)}{2}\int_0^L w_s \cdot w_s \, \mathrm{d}s \\ & + \frac{1}{2}\left[w_t(L,t) + \frac{\gamma L}{m_o} w_s(L,t) - \frac{\gamma}{2m_o} w(L,t)\right]^{\mathrm{T}} M_H \left[w_t(L,t) + \frac{\gamma L}{m_o} w_s(L,t) - \frac{\gamma}{2m_o} w(L,t)\right] \\ W_1 \leq\ & \frac{m_o + \gamma(L+1)}{2}\int_0^L w_t \cdot w_t \, \mathrm{d}s + \frac{B}{2}\int_0^L w_{ss} \cdot w_{ss} \, \mathrm{d}s + \frac{\lambda + \gamma(L+L^2)}{2}\int_0^L w_s \cdot w_s \, \mathrm{d}s \\ & + \frac{1}{2}\left[w_t(L,t) + \frac{\gamma L}{m_o} w_s(L,t) - \frac{\gamma}{2m_o} w(L,t)\right]^{\mathrm{T}} M_H \left[w_t(L,t) + \frac{\gamma L}{m_o} w_s(L,t) - \frac{\gamma}{2m_o} w(L,t)\right] \end{aligned} \tag{37}$$

Hence if we choose λ and γ such that

$$m_o - \gamma(L+1) = c_1, \quad \lambda - \gamma(L+L^2) = c_2 \tag{38}$$

where c_1 and c_2 are strictly positive constants, then the function W_1 defined in (35) is a proper function of $\int_0^L w_t \cdot w_t \, \mathrm{d}s$, $\int_0^L w_{ss} \cdot w_{ss} \, \mathrm{d}s$, $\int_0^L w_s \cdot w_s \, \mathrm{d}s$, and $[w_t(L,t) + (\gamma L/m_o) w_s(L,t) - (\gamma/2m_o) w(L,t)]$. We do not detail the conditions (38) at the moment, but deal with them after the boundary control I_H is designed since the constants λ and γ need to satisfy some other conditions later. It is noted that we do not include the riser displacement w, like $\int_0^L w \cdot w \, \mathrm{d}s$, in the function W_1 because this term causes difficulties in designing the control α_1 later. As such, after proof of convergence of $\int_0^L w_t \cdot w_t \, \mathrm{d}s$, $\int_0^L w_{ss} \cdot w_{ss} \, \mathrm{d}s$, and $\int_0^L w_s \cdot w_s \, \mathrm{d}s$, we will use Lemmas 2 and 3 in Appendix B to prove convergence of $\int_0^L w \cdot w \, \mathrm{d}s$ and the riserdisplacement w. Differentiating both sides of (35) with respect to t, along the solutions of the first four equations of the riser dynamics (24) results in

$$\begin{aligned} \dot{W}_1 =\ & \left(F \cdot w_t + B w_{ss} w_{st} + \lambda w_s \cdot w_t + \frac{\gamma F \cdot w_s s}{m_o} + \frac{\gamma w_t \cdot w_t s}{2} - \frac{\gamma F \cdot w}{2m_o}\right)\Bigg|_0^L - \lambda \int_0^L w_{ss} \cdot w_t \, \mathrm{d}s \\ & - \frac{\gamma B}{m_o}\int_0^L F \cdot w_{sss} s \, \mathrm{d}s - \frac{\gamma}{2m_o}\int_0^L F \cdot w_s \, \mathrm{d}s - \gamma \int_0^L w_t \cdot w_t \, \mathrm{d}s + \int_0^L q \cdot w_t \, \mathrm{d}s + \frac{\gamma}{m_o}\int_0^L q \cdot w_s s \, \mathrm{d}s \\ & - \frac{\gamma}{2m_o}\int_0^L q \cdot w \, \mathrm{d}s + \left(w_t(L,t) + \frac{\gamma L}{m_o} w_s(L,t) - \frac{\gamma}{2m_o} w(L,t)\right) \cdot \left(-B_H x_2 - F(L,t) + A_H x_3\right. \\ & \left. - \Delta(t, x_2) + \frac{\gamma L}{m_o} M_H w_{st}(L,t) - \frac{\gamma}{2m_o} M_H w_t(L,t)\right) \end{aligned} \tag{39}$$

where we have used (33). Now using (29), (30) and (28), and the boundary condition, see the last equation of (24), we can write (39) as

$$\begin{aligned} \dot{W}_1 \leq\ & \left(w_t(L,t) + \frac{\gamma L}{m_o} w_s(L,t) - \frac{\gamma}{2m_o} w(L,t)\right) \cdot \left(-B_H x_2 + (x_{3e} + \alpha_1) - \Delta(t, x_2)\right. \\ & \left. + \frac{\gamma L}{m_o} M_H w_{st}(L,t) - \frac{\gamma}{2m_o} M_H w_t(L,t)\right) + \lambda w_s(L,t) \cdot w_t(L,t) + \frac{\gamma L w_t(L,t) \cdot w_t(L,t)}{2} \\ & - \lambda \int_0^L w_{ss} \cdot w_t \, \mathrm{d}s - \frac{\gamma B}{4m_o}\int_0^L w_{ss} \cdot w_{ss} \, \mathrm{d}s - \frac{\gamma F(L,t) \cdot r_s(L,t)}{4m_o}\int_0^L w_s \cdot w_s \, \mathrm{d}s - \gamma \int_0^L w_t \cdot w_t \, \mathrm{d}s + \Omega_1 \end{aligned} \tag{40}$$

where we have used (27), and

$$\Omega_1 = \int_0^L \left(q \cdot w_t + \frac{\gamma}{m_o} q \cdot w_s s - \frac{\gamma}{2m_o} q \cdot w - \frac{\gamma}{4m_o} w_s \cdot w_s \int_s^L q(\sigma, t, w_t, r_\sigma(\sigma, t)) \cdot r_\sigma(\sigma, t) \, \mathrm{d}\sigma\right) \mathrm{d}s \tag{41}$$

From (40), we design the virtual control α_1 as follows:

$$\begin{aligned} \alpha_1 =\ & -K_1\left(w_t(L,t) + \frac{\gamma L}{m_o} w_s(L,t) - \frac{\gamma}{2m_o} w(L,t)\right) - \left(B_H + \frac{\gamma}{2m_o} M_H\right)\left(\frac{\gamma L}{m_o} w_s(L,t)\right. \\ & \left. - \frac{\gamma}{2m_o} w(L,t)\right) - \frac{\gamma L}{m_o} M_H w_{st}(L,t) + \hat{\Delta} + T_0 r_s(L,t) \end{aligned} \tag{42}$$

where K_1 is a positive definite diagonal matrix and T_0 is a positive scalar constant. The matrix K_1 and constant T_0 will be specified later. Inclusion of the term $T_0 r_s(L,t)$ in the virtual control α_1 is to provide sufficient tension in the riser. The term $\hat{\Delta}$ is an estimate of Δ, and is given by

$$\hat{\Delta} = -(\xi + Kx_2)$$

$$\dot{\xi} = -KM_H^{-1}\xi - K(\Phi + M_H^{-1}Kx_2) \tag{43}$$

where K is a diagonal positive definite matrix, and we have defined

$$\Phi = M_H^{-1}(-B_H x_2 - F(L,t) + A_H x_3) \tag{44}$$

Define the disturbance observer error as

$$\Delta_e = \Delta - \hat{\Delta} \tag{45}$$

Differentiating both sides of (45) along the solutions of (43) and the fourth equation of (24) gives

$$\dot{\Delta}_e = -KM_H^{-1}\Delta_e + \dot{\Delta} \tag{46}$$

This equation will be used in the stability analysis of the closed loop system after the control design is completed. It is noted that the disturbance observer (43) is based on Lemma 1 in [20] applied to the third equation of (24) with $\rho(x) = Kx$. The reader is also referred to [29] for an interesting application of the disturbance observer proposed in [20]. Since (40) contains the term $F(L,t) \cdot r_s(L,t)$, we need to find an expression for this term by substituting α_1 in (42) into the fourth equation of (24) to obtain

$$\begin{aligned} F(L,t) = & -M_H w_{tt}(L,t) - B_H w_t(L,t) - K_1\left(w_t(L,t) + \frac{\gamma L}{m_o} w_s(L,t) - \frac{\gamma}{2m_o} w(L,t)\right) \\ & - \left(B_H + \frac{\gamma}{2m_o}M_H\right)\left(\frac{\gamma L}{m_o} w_s(L,t) - \frac{\gamma}{2m_o}w(L,t)\right) - \frac{\gamma L}{m_o} M_H w_{st}(L,t) - \Delta_e + T_0 r_s(L,t) + x_{3e} \end{aligned} \tag{47}$$

Producting vector both sides of (47) with $r_s(L,t)$ gives

$$\begin{aligned} F(L,t)\cdot r_s(L,t) = & -r_s^{\mathrm{T}}(L,t)\left(K_1 + B_H + \frac{\gamma}{2m_o}M_H\right)\left(\frac{\gamma L}{m_o} w_s(L,t) - \frac{\gamma}{2m_o}w(L,t)\right) - \Delta_e \cdot r_s(L,t) \\ & + T_0 + x_{3e}\cdot r_s(L,t) \end{aligned} \tag{48}$$

where we have used $w_{tt}(L,t)\cdot r_s(L,t) = 0$ and $w_t(L,t)\cdot r_s(L,t)$ and $w_{st}(L,t)\cdot r_s(L,t) = 0$. Now substituting (42) and (48) into (40) and using completion of squares give

$$\begin{aligned} \dot{W}_1 \le & -c_3\left(w_t(L,t) + \frac{\gamma L}{m_o}w_s(L,t) - \frac{\gamma}{2m_o}w(L,t)\right)\cdot\left(w_t(L,t) + \frac{\gamma L}{m_o}w_s(L,t) - \frac{\gamma}{2m_o}w(L,t)\right) \\ & - c_4 w_t(L,t)\cdot w_t(L,t) - c_5 w_s(L,t)\cdot w_s(L,t) - c_6 w(L,t)\cdot w(L,t) - c_7\int_0^L w_t\cdot w_t\,\mathrm{d}s \\ & - c_8\int_0^L w_s\cdot w_s\,\mathrm{d}s - c_9\int_0^L w_{ss}\cdot w_{ss}\,\mathrm{d}s + \left(w_t(L,t) + \frac{\gamma L}{m_o}w_s(L,t) - \frac{\gamma}{2m_o}w(L,t)\right)(x_{3e} - \Delta_e) \\ & + \left(\frac{\gamma\Delta_e\cdot r_s(L,t)}{4m_o} - \frac{\gamma x_{3e}\cdot r_s(L,t)}{4m_o}\right)\int_0^L w_s\cdot w_s\,\mathrm{d}s + \Omega_1 \end{aligned} \tag{49}$$

where

$$c_3 = \lambda_{\min}(A) - \varepsilon_0 \quad \text{with } A = K_1 + B_H + \frac{\gamma}{2m_o}M_H$$

$$c_4 = \varepsilon_0 - \frac{\gamma L}{2} - \frac{2\varepsilon_0\varepsilon_1\gamma L}{m_o} - \frac{\varepsilon_0\varepsilon_2\gamma}{m_o} - \lambda\varepsilon_3$$

$$c_5 = \frac{\varepsilon_0\gamma^2 L^2}{m_o^2}, \quad c_6 = \frac{\varepsilon_0\gamma^2}{4m_o^2}, \quad c_7 = \gamma - \lambda\varepsilon_5$$

$$c_8 = \frac{\gamma T_0}{2m_o} - \frac{\gamma^2\lambda_{\max}(A)L}{4m_o^2} - \frac{\sqrt{2}\gamma^2\lambda_{\max}(A)L}{8m_o^2} - \frac{\lambda}{4\varepsilon_3} - \frac{\gamma^2\varepsilon_0 L}{2m_o^2} - \left(\frac{\gamma\varepsilon_0}{4\varepsilon_2 m_o} + \frac{\gamma^2\varepsilon_0 L}{2m_o^2}\right)(4L^2+1)$$

$$c_9 = \frac{\gamma B}{4m_o} - \frac{\lambda}{4\varepsilon_3} - \frac{\lambda}{4\varepsilon_5} - \frac{\gamma^2\varepsilon_0 L}{2m_o^2} - \frac{\gamma T_0}{4\varepsilon_4 m_o} \tag{50a}$$

with $\varepsilon_i,\ i=0,\ldots,5$ being positive constants, and $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ the minimum and maximum eigenvalue of the matrix A, respectively. The positive constants $\varepsilon_i,\ i=0,\ldots,5$ and γ are picked such that $c_i,\ i=1,\ldots,9$, where c_1 and c_2 are given in (38) are strictly positive. Now substituting the virtual control α_1 given in (42) into the fourth equation of (24) give the first sub-closed loop system:

$$\begin{aligned}\dot{x}_2 = M_H^{-1}\Big(&-B_H x_2 - K_1\left(w_t(L,t)+\frac{\gamma L}{m_o}w_s(L,t)-\frac{\gamma}{2m_o}w(L,t)\right)-\left(B_H+\frac{\gamma}{2m_o}M_H\right)\\ &\times\left(\frac{\gamma L}{m_o}w_s(L,t)-\frac{\gamma}{2m_o}w(L,t)\right)-\frac{\gamma L}{m_o}M_H w_{st}(L,t)+T_0 r_s(L,t)-\Delta_e+x_{3e}\Big)\end{aligned}\tag{50b}$$

To prepare for the next step, let us calculate $\dot{x}_{3e}$. Differentiating both sides of (34) along the solutions of (42) and the fourth equation of (24) with a note that the virtual control α_1 is a smooth function of $w(L,t), w_t(L,t), w_s(L,t), w_{st}(L,t), r_s(L,t)$ and $\hat{\Delta}$ results in

$$\begin{aligned}\dot{x}_{3e} = {}& A_H\bar{V}_H^{-1}(-A_H x_2 - C_{HT}x_3+\Psi x_4)-\frac{\partial\alpha_1}{\partial w(L,t)}w_t(L,t)-\frac{\partial\alpha_1}{\partial w_s(L,t)}w_{st(L,t)}\\ &-\frac{\partial\alpha_1}{\partial w_{st}(L,t)}w_{stt}(L,t)-\frac{\partial\alpha_1}{\partial r_s(L,t)}r_{st}(L,t)-\frac{\partial\alpha_1}{\partial\hat{\Delta}}(KM_H^{-1}\xi+K(\Phi+KM_H^{-1}x_2))\\ &-\left(\frac{\partial\alpha_1}{\partial x_2}-\frac{\partial\alpha_1}{\partial\hat{\Delta}}K\right)M_H^{-1}(-B_H x_2-F(L,t)+A_H x_3-\Delta(t,x_2))\end{aligned}\tag{51}$$

3.2. Step 2

Our goal at this step is to regulate x_{3e} to a small neighborhood of the origin by considering the fourth equation of the entire system (24) where for simplicity of the design process, we consider Ψx_4 as a control instead of x_4. As such, we define

$$x_{4e} = \Psi x_4 - \alpha_2 \tag{52}$$

where α_2 is a virtual control of Ψx_4. To design the virtual control α_2, we consider the following Lyapunov function candidate:

$$W_2 = W_1 + \tfrac{1}{2}x_{3e}^{\mathrm{T}}x_{3e} \tag{53}$$

whose derivative along the solutions of (49) and (51) is

$$\begin{aligned}\dot{W}_2 \le {}& -c_3\left(w_t(L,t)+\frac{\gamma L}{m_o}w_s(L,t)-\frac{\gamma}{2m_o}w(L,t)\right)\cdot\left(w_t(L,t)+\frac{\gamma L}{m_o}w_s(L,t)-\frac{\gamma}{2m_o}w(L,t)\right)\\ &-c_4 w_t(L,t)\cdot w_t(L,t)-c_5 w_s(L,t)\cdot w_s(L,t)-c_6 w(L,t)\cdot w(L,t)-c_7\int_0^L w_t\cdot w_t\,\mathrm{d}s\\ &-c_8\int_0^L w_s\cdot w_s\,\mathrm{d}s-c_9\int_0^L w_{ss}\cdot w_{ss}\,\mathrm{d}s+\left(w_t(L,t)+\frac{\gamma L}{m_o}w_s(L,t)-\frac{\gamma}{2m_o}w(L,t)\right)(x_{3e}-\Delta_e)\\ &+x_{3e}^{\mathrm{T}}\Big[A_H\bar{V}_H^{-1}(-A_H x_2-C_{HT}x_3+\alpha_2+x_{4e})-\frac{\partial\alpha_1}{\partial w(L,t)}w_t(L,t)-\frac{\partial\alpha_1}{\partial w_s(L,t)}w_{st(L,t)}\\ &-\frac{\partial\alpha_1}{\partial w_{st}(L,t)}w_{stt}(L,t)-\frac{\partial\alpha_1}{\partial r_s(L,t)}r_{st}(L,t)-\frac{\partial\alpha_1}{\partial\hat{\Delta}}(KM_H^{-1}\xi+K(\Phi+KM_H^{-1}x_2))\\ &-\left(\frac{\partial\alpha_1}{\partial x_2}-\frac{\partial\alpha_1}{\partial\hat{\Delta}}K\right)M_H^{-1}(-B_H x_2-F(L,t)+A_H x_3-\hat{\Delta}-\Delta_e)\Big]\\ &+\left(\frac{\gamma\Delta_e.r_s(L,t)}{4m_o}-\frac{\gamma x_{3e}.r_s(L,t)}{4m_o}\right)\int_0^L w_s.w_s\,\mathrm{d}s+\Omega_1\end{aligned}\tag{54}$$

which suggests that we choose the virtual control α_2 as follows:

$$\begin{aligned}\alpha_2 = (A_H\bar{V}_H^{-1})^{-1}\Big[&-\left(w_t(L,t)+\frac{\gamma L}{m_o}w_s(L,t)-\frac{\gamma}{2m_o}w(L,t)\right)+A_H\bar{V}_H^{-1}(A_H x_2+C_{HT}x_3)\\ &+\frac{\partial\alpha_1}{\partial w(L,t)}w_t(L,t)+\frac{\partial\alpha_1}{\partial w_s(L,t)}w_{st(L,t)}+\frac{\partial\alpha_1}{\partial w_{st}(L,t)}w_{stt}(L,t)+\frac{\partial\alpha_1}{\partial r_s(L,t)}r_{st}(L,t)\\ &+\frac{\partial\alpha_1}{\partial\hat{\Delta}}(KM_H^{-1}\xi+K(\Phi+KM_H^{-1}x_2))+\left(\frac{\partial\alpha_1}{\partial x_2}-\frac{\partial\alpha_1}{\partial\hat{\Delta}}K\right)M_H^{-1}(-B_H x_2-F(L,t)+A_H x_3-\hat{\Delta})-K_2 x_{3e}\Big]\end{aligned}\tag{55}$$

where K_2 is a diagonal positive definite matrix. It should be noted that unlike standard backstepping technique, we do not use the virtual control α_2 to cancel the term $-(\gamma x_{3e}\cdot r_s(L,t)/4m_o)\int_0^L w_s\cdot w_s\,\mathrm{d}s$ in (54) since it requires measurement of $w_s(s,t)$ along the riser. As such, this term will be dominated by the terms $-c_8\int_0^L w_s w_s\,\mathrm{d}s$ and $-x_{3e}^{\mathrm{T}}K_2 x_{3e}$, see (56). Noticing that the virtual control α_2 is a smooth function of $w(L,t)$, $w_t(L,t)$, $w_s(L,t)$, $w_{st}(L,t)$, $r_s(L,t)$, $w_{stt}(L,t)$, x_2, x_3, ξ, $\hat{\Delta}$ and $F(L,t)$.

Substituting (55) into (54) results in

$$\begin{aligned}\dot{W}_2 \le & -c_3\left(w_t(L,t)+\frac{\gamma L}{m_o}w_s(L,t)-\frac{\gamma}{2m_o}w(L,t)\right)\cdot\left(w_t(L,t)+\frac{\gamma L}{m_o}w_s(L,t)-\frac{\gamma}{2m_o}w(L,t)\right)\\ & -c_4w_t(L,t)\cdot w_t(L,t)-c_5w_s(L,t)\cdot w_s(L,t)-c_6w(L,t)\cdot w(L,t)-c_7\int_0^L w_t\cdot w_t\,\mathrm{d}s\\ & -c_8\int_0^L w_s\cdot w_s\,\mathrm{d}s-c_9\int_0^L w_{ss}\cdot w_{ss}\,\mathrm{d}s-x_{3e}^{\mathrm{T}}K_2x_{3e}+x_{3e}^{\mathrm{T}}A_H\bar{V}_H^{-1}x_{4e}\\ & -\left(w_t(L,t)+\frac{\gamma L}{m_o}w_s(L,t)-\frac{\gamma}{2m_o}w(L,t)\right)\Delta_e+x_{3e}^{\mathrm{T}}\left(\frac{\partial\alpha_1}{\partial x_2}-\frac{\partial\alpha_1}{\partial\hat{\Delta}}K\right)M_H^{-1}\Delta_e\\ & +\left(\frac{\gamma\Delta_e\cdot r_s(L,t)}{4m_o}-\frac{\gamma x_{3e}\cdot r_s(L,t)}{4m_o}\right)\int_0^L w_s\cdot w_s\,\mathrm{d}s+\Omega_1\end{aligned}\tag{56}$$

On the other hand, substituting the virtual control α_2 into (51) gives the second sub-closed loop system

$$\dot{x}_{3e}=-w_t(L,t)-\frac{\gamma L}{m_o}w_s(L,t)+\frac{\gamma}{2m_o}w(L,t)-K_2x_{3e}+A_H\bar{V}_H^{-1}x_{4e}-\left(\frac{\partial\alpha_1}{\partial x_2}-\frac{\partial\alpha_1}{\partial\hat{\Delta}}K\right)M_H^{-1}\Delta_e\tag{57}$$

To prepare for the next step, let us calculate $\dot{x}_{4e}$. Differentiating both sides of (52) along the solutions of the fifth equation of (24) and (55) gives

$$\begin{aligned}\dot{x}_{4e}= & \frac{\partial\Psi}{\partial x_3}\bar{V}_H^{-1}(-A_Hx_2-C_{HT}x_3+\Psi x_4)+\left(\frac{\partial\Psi}{\partial x_4}+\Psi\right)T_{Hv}^{-1}(-x_4+K_{Hv}I_H)-\frac{\partial\alpha_2}{\partial w(L,t)}w_t(L,t)\\ & -\frac{\partial\alpha_2}{\partial w_s(L,t)}w_{st}(L,t)-\frac{\partial\alpha_2}{\partial w_{stt}(L,t)}w_{sttt}(L,t)-\frac{\partial\alpha_2}{\partial r_s(L,t)}r_{st}(L,t)-\frac{\partial\alpha_2}{\partial x_3}\bar{V}_H^{-1}(-A_Hx_2-C_{HT}x_3\\ & +\Psi x_4)-\frac{\partial\alpha_2}{\partial\xi}\dot{\xi}-\frac{\partial\alpha_2}{\partial F(L,t)}\dot{F}(L,t)+\frac{\partial\alpha_2}{\partial\hat{\Delta}}\xi+\left(\frac{\partial\alpha_2}{\partial\hat{\Delta}}K-\frac{\partial\alpha_2}{\partial x_2}\right)M_H^{-1}(-B_Hx_2-F(L,t)+A_Hx_3-\Delta(t,x_2))\end{aligned}\tag{58}$$

3.3. Step 3

This is the final step. The actual control input I_H will be designed to regulate x_{4e} to a small neighborhood of the origin. To design the actual control input I_H, we consider the following Lyapunov function candidate:

$$W_3=W_2+\tfrac{1}{2}x_{4e}^{\mathrm{T}}x_{4e}\tag{59}$$

whose derivative along the solutions of (58) and (56) is

$$\begin{aligned}\dot{W}_3\le & -c_3\left(w_t(L,t)+\frac{\gamma L}{m_o}w_s(L,t)-\frac{\gamma}{2m_o}w(L,t)\right)\cdot\left(w_t(L,t)+\frac{\gamma L}{m_o}w_s(L,t)-\frac{\gamma}{2m_o}w(L,t)\right)\\ & -c_4w_t(L,t)\cdot w_t(L,t)-c_5w_s(L,t)\cdot w_s(L,t)-c_6w(L,t)\cdot w(L,t)-c_7\int_0^L w_t\cdot w_t\,\mathrm{d}s\\ & -c_8\int_0^L w_s\cdot w_s\,\mathrm{d}s-c_9\int_0^L w_{ss}\cdot w_{ss}\,\mathrm{d}s-x_{3e}^{\mathrm{T}}K_2x_{3e}+x_{4e}^{\mathrm{T}}\Bigg[A_H\bar{V}_H^{-1}x_{3e}+\frac{\partial\Psi}{\partial x_3}\bar{V}_H^{-1}(-A_Hx_2\\ & -C_{HT}x_3+\Psi x_4)+\left(\frac{\partial\Psi}{\partial x_4}+\Psi\right)T_{Hv}^{-1}(-x_4+K_{Hv}I_H)-\frac{\partial\alpha_2}{\partial w(L,t)}w_t(L,t)-\frac{\partial\alpha_2}{\partial w_s(L,t)}w_{st}(L,t)\\ & -\frac{\partial\alpha_2}{\partial w_{stt}(L,t)}w_{sttt}(L,t)-\frac{\partial\alpha_2}{\partial r_s(L,t)}r_{st}(L,t)-\frac{\partial\alpha_2}{\partial x_3}\bar{V}_H^{-1}(-A_Hx_2-C_{HT}x_3+\Psi x_4)-\frac{\partial\alpha_2}{\partial\xi}\dot{\xi}\\ & -\frac{\partial\alpha_2}{\partial F(L,t)}\dot{F}(L,t)+\frac{\partial\alpha_2}{\partial\hat{\Delta}}\xi+\left(\frac{\partial\alpha_2}{\partial\hat{\Delta}}K-\frac{\partial\alpha_2}{\partial x_2}\right)M_H^{-1}(-B_Hx_2-F(L,t)+A_Hx_3-\hat{\Delta})\Bigg]\\ & -x_{4e}^{\mathrm{T}}\left(\frac{\partial\alpha_2}{\partial\hat{\Delta}}K-\frac{\partial\alpha_2}{\partial x_2}\right)M_H^{-1}\Delta_e-\left(w_t(L,t)+\frac{\gamma L}{m_o}w_s(L,t)-\frac{\gamma}{2m_o}w(L,t)\right)\Delta_e\\ & +x_{3e}^{\mathrm{T}}\left(\frac{\partial\alpha_1}{\partial x_2}-\frac{\partial\alpha_1}{\partial\hat{\Delta}}K\right)M_H^{-1}\Delta_e+\left(\frac{\gamma\Delta_e.r_s(L,t)}{4m_o}-\frac{\gamma x_{3e}.r_s(L,t)}{4m_o}\right)\int_0^L w_s.w_s\,\mathrm{d}s+\Omega_1\end{aligned}\tag{60}$$

which suggests that we choose the actual control I_H as

$$\begin{aligned}I_H= & x_4+\left[\left(\frac{\partial\Psi}{\partial x_4}+\Psi\right)T_{Hv}^{-1}K_{Hv}\right]^{-1}\Bigg[-A_H\bar{V}_H^{-1}x_{3e}-\frac{\partial\Psi}{\partial x_3}\bar{V}_H^{-1}(-A_Hx_2-C_{HT}x_3+\Psi x_4)\\ & -K_3x_{4e}+\frac{\partial\alpha_2}{\partial w(L,t)}w_t(L,t)+\frac{\partial\alpha_2}{\partial w_s(L,t)}w_{st}(L,t)+\frac{\partial\alpha_2}{\partial w_{stt}(L,t)}w_{sttt}(L,t)\end{aligned}$$

$$+\frac{\partial\alpha_2}{\partial r_s(L,t)}r_{st}(L,t)+\frac{\partial\alpha_2}{\partial x_3}\bar{V}_H^{-1}(-A_Hx_2-C_{HT}x_3+\Psi x_4)+\frac{\partial\alpha_2}{\partial\xi}\dot{\xi}+\frac{\partial\alpha_2}{\partial F(L,t)}\dot{F}(L,t)$$
$$+\frac{\partial\alpha_2}{\partial\hat{\Delta}}\xi-\left(\frac{\partial\alpha_2}{\partial\hat{\Delta}}K-\frac{\partial\alpha_2}{\partial x_2}\right)M_H^{-1}(-B_Hx_2-F(L,t)+A_Hx_3-\hat{\Delta})\Bigg] \tag{61}$$

where K_3 is a diagonal positive definite matrix. Substituting (61) into (58) gives the third sub-closed loop system:

$$\dot{x}_{4e}=-A_H\bar{V}_H^{-1}x_{3e}-K_3x_{4e}-\left(\frac{\partial\alpha_2}{\partial\hat{\Delta}}K-\frac{\partial\alpha_2}{\partial x_2}\right)M_H^{-1}\Delta_e \tag{62}$$

Now substituting (61) into (60) gives

$$\begin{aligned}\dot{W}_3\leq\ &-c_3\left(w_t(L,t)+\frac{\gamma L}{m_o}w_s(L,t)-\frac{\gamma}{2m_o}w(L,t)\right)\cdot\left(w_t(L,t)+\frac{\gamma L}{m_o}w_s(L,t)-\frac{\gamma}{2m_o}w(L,t)\right)\\&-c_4w_t(L,t)\cdot w_t(L,t)-c_5w_s(L,t)\cdot w_s(L,t)-c_6w(L,t)\cdot w(L,t)-c_7\int_0^L w_t\cdot w_t\,\mathrm{d}s\\&-c_8\int_0^L w_s\cdot w_s\,\mathrm{d}s-c_9\int_0^L w_{ss}\cdot w_{ss}\,\mathrm{d}s-x_{3e}^{\mathrm{T}}K_2x_{3e}-x_{4e}^{\mathrm{T}}K_3x_{4e}\\&-x_{4e}^{\mathrm{T}}\left(\frac{\partial\alpha_2}{\partial\hat{\Delta}}K-\frac{\partial\alpha_2}{\partial x_2}\right)M_H^{-1}\Delta_e-\left(w_t(L,t)+\frac{\gamma L}{m_o}w_s(L,t)-\frac{\gamma}{2m_o}w(L,t)\right)\Delta_e\\&+x_{3e}^{\mathrm{T}}\left(\frac{\partial\alpha_1}{\partial x_2}-\frac{\partial\alpha_1}{\partial\hat{\Delta}}K\right)M_H^{-1}\Delta_e+\left(\frac{\gamma\Delta_e.r_s(L,t)}{4m_o}-\frac{\gamma x_{3e}.r_s(L,t)}{4m_o}\right)\int_0^L w_s\cdot w_s\,\mathrm{d}s+\Omega_1\end{aligned} \tag{63}$$

For convenience of stability analysis, which will be carried out in Appendix C, we rewrite the closed loop system consisting of (46), (50b), (57), (62) and the first three equations and the last equation of (24) as follows:

$$\begin{aligned}&m_ow_{tt}=F_s+q,\quad s\in(0,L)\\&r_s\times(Bw_{sss}+F)=0,\quad s\in(0,L)\\&\dot{x}_1=x_2\\&\dot{x}_2=M_H^{-1}\left(-B_Hx_2-K_1\left(w_t(L,t)+\frac{\gamma L}{m_o}w_s(L,t)-\frac{\gamma}{2m_o}w(L,t)\right)-\left(B_H+\frac{\gamma}{2m_o}M_H\right)\right.\\&\qquad\left.\times\left(\frac{\gamma L}{m_o}w_s(L,t)-\frac{\gamma}{2m_o}w(L,t)\right)-\frac{\gamma L}{m_o}M_Hw_{st}(L,t)+T_0r_s(L,t)-\Delta_e+x_{3e}\right)\\&\dot{x}_{3e}=-w_t(L,t)-\frac{\gamma L}{m_o}w_s(L,t)+\frac{\gamma}{2m_o}w(L,t)-K_2x_{3e}+A_H\bar{V}_H^{-1}x_{4e}-\left(\frac{\partial\alpha_1}{\partial x_2}-\frac{\partial\alpha_1}{\partial\hat{\Delta}}K\right)M_H^{-1}\Delta_e\\&\dot{x}_{4e}=-A_H\bar{V}_H^{-1}x_{3e}-K_3x_{4e}-\left(\frac{\partial\alpha_2}{\partial\hat{\Delta}}K-\frac{\partial\alpha_2}{\partial x_2}\right)M_H^{-1}\Delta_e\\&\dot{\Delta}_e=-\frac{k}{m_H}\Delta_e+\dot{\Delta}\\&w(0,t)=0,\quad w_{ss}(0,t)=0,\quad w_{ss}(L,t)=0\end{aligned} \tag{64}$$

Theorem 1. *Under* Assumption 1, *the control input I_H given in* (61) *solves the control objective provided that the design constants γ and K_1 are chosen such that the conditions given in* (38) *and* (50a) *hold. In particular, the solutions of the closed loop system* (64) *exist and are unique. Moreover, when the external disturbance vector q is zero and and the disturbance $\Delta(t,w_t(L,t))$ is constant, all the terms $\|w(s,t)\|$, $\int_0^L w_s(s,t)\cdot w_s(s,t)\,\mathrm{d}s$, $\int_0^L w_t(s,t)\cdot w_t(s,t)\,\mathrm{d}s$ and $\int_0^L w_{ss}(s,t)\cdot w_{ss}(s,t)\,\mathrm{d}s$ exponentially converge to zero for all $s\in[0,L]$ and $t\geq t_0$, and when the external disturbance vector q is different from zero but bounded and the disturbance $\Delta(t,w_t(L,t))$ is time-varying with bounded derivative with respect to time, all the terms $\|w(s,t)\|$, $\int_0^L w_s(s,t)\cdot w_s(s,t)\,\mathrm{d}s$, $\int_0^L w_t(s,t)\cdot w_t(s,t)\,\mathrm{d}s$ and $\int_0^L w_{ss}(s,t)\cdot w_{ss}(s,t)\,\mathrm{d}s$ exponentially converge to some small positive constants for all $s\in[0,L]$ andt $\geq t_0$.*

Proof. See Appendix C.

4. Simulations

In this section, we carry out some numerical simulations to illustrate the effectiveness of the proposed boundary controller. We take identical three hydraulic systems with the parameters based on [32] as follows: $m_{iH}=1000\,\mathrm{kg}$,

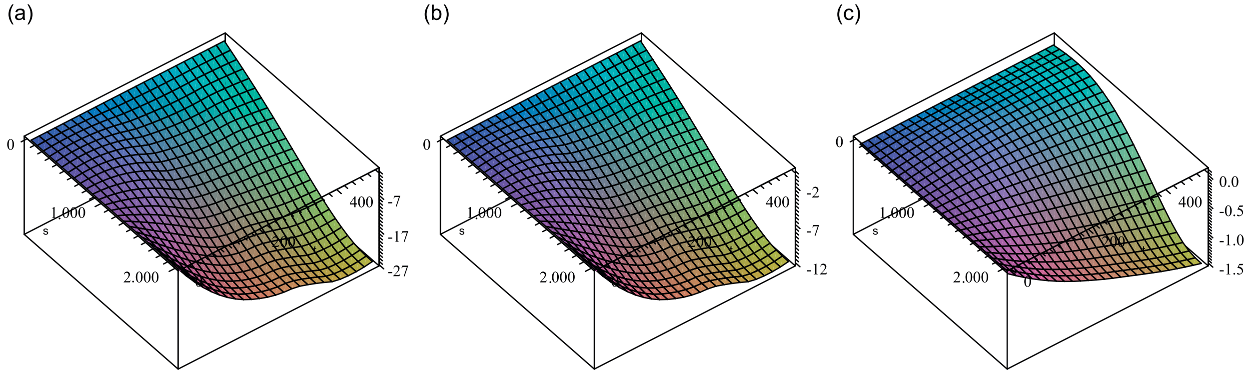

Fig. 2. Simulation results without control: displacements w^x, w^y, w^z. (a) w_x; (b) w_y; (c) w_z.

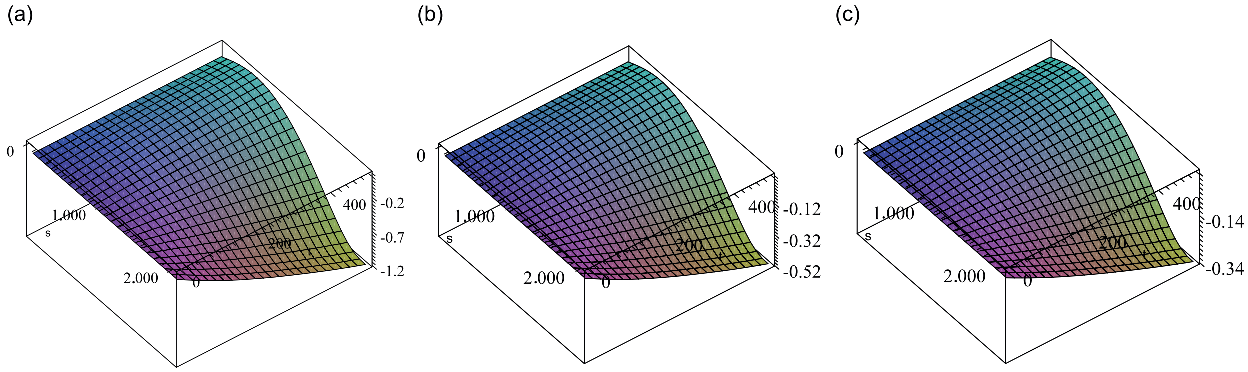

Fig. 3. Simulation results with control: displacements w^x, w^y, w^z. (a) w_x; (b) w_y; (c) w_z.

$A_{iH} = 0.65\,\mathrm{m}^2$, $b_{iH} = 40\,\mathrm{N/(m/s)}$, $4\beta_{iHe}/V_{iH} = 4.53 \times 10^8\,\mathrm{N/m}^5$, $C_{iHD} = 2.21 \times 10^{-14}\,\mathrm{m}^5/\mathrm{N\,s}$, $C_{iHD}W_{iH}/\sqrt{\rho_i} = 3.42 \times 10^{-5}\,\mathrm{m}^3\sqrt{\mathrm{N}}\,\mathrm{s}$, $P_{iHS} = 10\,342\,500\,\mathrm{Pa}$, $k_{iHv} = 0.0324$, and $\tau_{iHv} = 0.00636$, for $i = 1,2,3$. The riser parameters are taken from [2] as follows: length $L = 1000\,\mathrm{m}$, diameter $D = 0.61\,\mathrm{m}$, density $\rho_r = 1250\,\mathrm{kg/m}^3$, Young's modulus $E = 2 \times 10^{10}\,\mathrm{kg/m}$. The parameters of the distributed damping and external forces are taken as follows: $C_{\mathrm{LD}} = 0.7$, $C_{\mathrm{LD}} = 0.35$, $D_H = 0.87\,\mathrm{m}$, $\rho_w = 1025\,\mathrm{kg/m}^3$, and $w_e = 1.132\,\mathrm{KN/m}$. We assume that the disturbance $\Delta(t, w_t(L,t)) = 0.5\,\mathrm{diag}(m_{1H}\sin(0.5t + 2\pi\,\mathrm{rand}()), m_{2H}\cos(0.5t + 2\pi\,\mathrm{rand}()), m_{3H}\sin(0.2t + 2\pi\,\mathrm{rand}()))$ with rand( ) is a number between 0 and 1. The initial conditions are taken as $w(s,t_0) = [0,0,0]^{\mathrm{T}}$, $w_t(s,t_0) = [0,0,0]^{\mathrm{T}}$, $\xi(t_0) = [0,0,0]^{\mathrm{T}}$, $x_3(t_0) = [0,0,0]^{\mathrm{T}}, x_4(t_0) = [0,0,0]^{\mathrm{T}}$. The observer and control gains are chosen as follows: $K = \mathrm{diag}(2,2,2)$, $K_1 = \mathrm{diag}(4,4,4)$, $K_2 = \mathrm{diag}(6,6,6)$, $K_3 = \mathrm{diag}(10,10,10)$, and $T_0 = 3.5 \times 10^6$. It is directly checked that the chosen observer and control gains satisfy the required conditions given in (38) and (50). The ocean current velocity vector is assumed to be generated from wind at the ocean surface and dropped to zero at the sea bed [33]: $V = [(1/L)s, (0.5/L)s, 0]^{\mathrm{T}}$. We run simulations without the proposed boundary controller, i.e. $K_1 = \mathrm{diag}(0,0,0)$, $K_2 = \mathrm{diag}(0,0,0)$, and $K_3 = \mathrm{diag}(0,0,0)$, and with the proposed boundary controller, i.e. $K_1 = \mathrm{diag}(4,4,4)$, $K_2 = \mathrm{diag}(6,6,6)$, and $K_3 = \mathrm{diag}(10,10,10)$. The length of simulation time for both cases is 500 s. Displacements $w = [w^x, w^y, w^z]^{\mathrm{T}}$ for the uncontrolled and controlled cases are displayed in Figs. 2 and 3, respectively. In Fig. 4, the error signals of the system along the x-axis are plotted. It is seen from these figures that the proposed boundary controller can reduce deflections of the riser in all directions (x,y,z) significantly, i.e. the displacement magnitudes are significantly reduced. For example, in the x direction, the displacement magnitudereduces from 27 to 1.2 m at the top end of the riser. This illustrates the effectiveness of the proposed boundary controller in the sense that it is able to drive the riser to the small neighborhood of its equilibrium position.

5. Conclusions

The equations of motion of a marine riser-hydraulic system were presented. These equations were then used for the design of the boundary controller at the top end of the riser based on Lyapunov's direct method. The proposed controller robustly stabilized the riser at its equilibrium vertical position. Proof of existence and uniqueness of the solutions of the closed loop system was given. The keys of the paper are the proposed Lyapunov function candidate (35) and various properties of the riser dynamics given in Lemma 1. The rest of the paper requires a careful manipulation of integration by

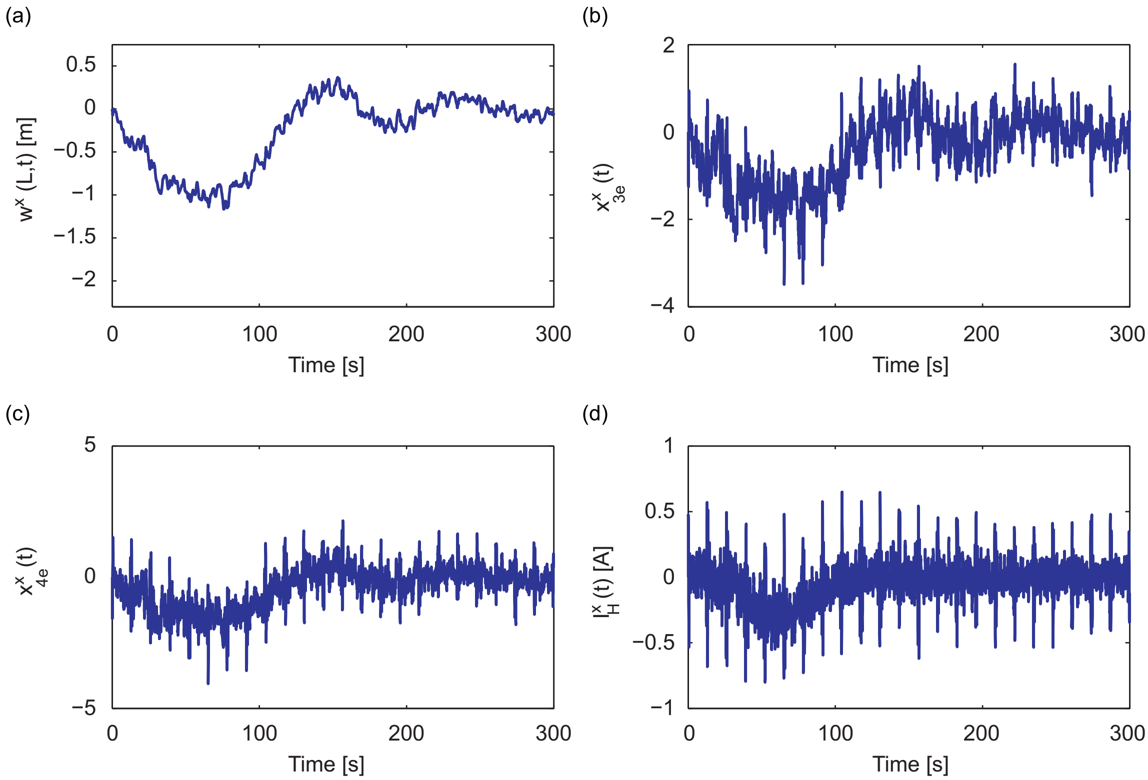

Fig. 4. Simulation result with the proposed controller: (a) transverse displacement at the top end $\eta(L,t)$; (b) virtual error $x_{3e}(t)$; (c) virtual error $x_{4e}(t)$; (d) control input $i_H(t)$.

parts and a proper use of Poincare's inequalities in bounding the derivatives of the Lyapunov function candidates W_2 and W_3. Future work focuses on relaxing items made in Assumption 1, and carrying out experiments to test the effectiveness of the proposed boundary controller. Particularly, an immediate task is to consider an arbitrarily initial position of the riser and to take the effect of the torsional moments into account in the boundary control design.

Appendix A. Proof of Lemma 1

We first prove (26). Since $r = r^o + w$ and the inextensible condition gives $r_s^o.r_s^o = 1$, we have

$$\left.\begin{array}{l} r_s\cdot r_s = (r_s^o + w_s)\cdot(r_s^o + w_s) = w_s\cdot w_s + 2r_s^o\cdot w_s + r_s^0\cdot r_s^o \\ r_s\cdot r_s = 1 \\ r_s^o\cdot r_s^o = 1 \\ w_s\cdot r_s = w_s\cdot(r_s^o + w_s) = w_s\cdot w_s + r_s^o\cdot w_s \end{array}\right\} \Rightarrow w_s\cdot r_s = \frac{1}{2}w_s\cdot w_s \tag{65}$$

The inequality (27) can be proved by noting that $w\cdot w \le r\cdot r + r^o\cdot r^o$ since the angle between r and r^o is never greater than $\pi/2$ due to the straight initial condition. Eq. (28) is easily proved by crossing vector both sides of the second equation of (24) with w_{ss}, and noticing the last equation of (24). Similarly, Eq. (29) can be proved by crossing vector both sides of the second equation of (24) with w_s, and noticing the last equation of (24). Eq. (30) can be proved by adding both sides of (28) with $F_s\cdot r_s$ then integrating both sides of the resulting equation from s to L, and noting the inextensible condition implies that $w_{tt}\cdot r_s = 0,\ \forall t\in\mathbb{R}^+, s\in[0,L]$. Eq. (31) can be proved by crossing vector both sides of the second equation of (24) with w_{tt} plus a note that $w_{tt}\cdot r_s = 0,\ \forall (s,t)\in((0,L),\mathbb{R}^+)$ due to the inextensible condition. To prove (32), we consider

$$(r_s(s,t)\cdot w(s,t))_s = r_{ss}(s,t)\cdot w(s,t) + r_s(s,t)\cdot w_s(s,t) = w_{ss}(s,t)\cdot w(s,t) + \tfrac{1}{2}w_s(s,t)\cdot w_s(s,t) \tag{66}$$

Integrating both sides of (66) from 0 to L, and noting the last equation of (24) give (32). Eq. (33) can be proved by crossing vector both sides of the second equation of (24) with $w_{st}(s,t)$, then producting both sides of the resulting equation with r_s and noticing that $r_{st}\cdot r_s = w_{st}\cdot r_s = 0$ due to the straight initial condition and $r_s\cdot r_s = 1$.

Appendix B. Simplified Poincare inequalities

Lemma 2. *For any* $y=[y_1,\ldots,y_i,\ldots,y_n]^{\mathrm{T}}$ *with* $y_i\in C^1[0,L]$, $i=1,\ldots,n$, *the following inequalities hold*:

$$\int_0^L y(s)\cdot y(s)\,\mathrm{d}s \le 2Ly(0)\cdot y(0)+4L^2\int_0^L y_s(s)\cdot y_s(s)\,\mathrm{d}s \tag{67}$$

$$\int_0^L y(s)\cdot y(s)\,\mathrm{d}s \le 2Ly(L)\cdot y(L)+4L^2\int_0^L y_s(s)\cdot y_s(s)\,\mathrm{d}s \tag{68}$$

Proof. We prove (68). The proof of (67) is similar by using a change of coordinate $\xi=L-s$. Using integration by parts, we have

$$\begin{aligned}\int_0^L y(s).y(s)\,\mathrm{d}s &= y(s)\cdot y(s)s|_0^L-2\int_0^L sy(s)\cdot y_s(s)\,\mathrm{d}s\\ &\le Ly(L)\cdot y(L)+\frac{1}{2}\int_0^L y(s)\cdot y(s)\,\mathrm{d}s+2\int_0^L s^2y_s(s)\cdot y_s(s)\,\mathrm{d}s\\ &\le Ly(L)\cdot y(L)+\frac{1}{2}\int_0^L y(s)\cdot y(s)\,\mathrm{d}s+2L^2\int_0^L y_s(s)\cdot y_s(s)\,\mathrm{d}s\end{aligned} \tag{69}$$

which gives (68). □

Lemma 3. *For any* $y=[y_1,\ldots,y_i,\ldots,y_n]^{\mathrm{T}}$ *with* $y_i\in C^1[0,L]$, $i=1,\ldots,n$, *the following inequalities hold*:

$$\max_{s\in[0,L]}(y(s)\cdot y(s))\le y(0)\cdot y(0)+2\sqrt{\int_0^L y(s)\cdot y(s)\,\mathrm{d}s}\sqrt{\int_0^L y_s(s)\cdot y_s(s)\,\mathrm{d}s} \tag{70}$$

$$\max_{s\in[0,L]}(y(s)\cdot y(s))\le y(L)\cdot y(L)+2\sqrt{\int_0^L y(s)\cdot y(s)\,\mathrm{d}s}\sqrt{\int_0^L y_s(s).y_s(s)\,\mathrm{d}s} \tag{71}$$

Proof. We prove (70). The proof of (71) is similar by using a change of coordinate $\xi=L-s$. From fundamental of calculus, we have

$$\begin{aligned}y(s)\cdot y(s) &= y(0)\cdot y(0)+2\int_0^s y(\zeta)\cdot y_\zeta(\zeta)\,\mathrm{d}\zeta\\ &\le y(0)\cdot y(0)+2\sqrt{\int_0^s y(\zeta)\cdot y(\zeta)\,\mathrm{d}\zeta}\sqrt{\int_0^s y_\zeta(\zeta)\cdot y_\zeta(\zeta)\,\mathrm{d}\zeta}\end{aligned} \tag{72}$$

where we have used the Cauchy–Schwartz inequality. □

Appendix C. Proof of Theorem 1

C.1. Proof of existence and uniqueness

Let $H^2(0,L)$ be the usual Hilbert space [34]. Our analysis is based on the Sobolev spaces:

$$V_S=w\in H^2(0,L)|w(0,t)=0 \tag{73}$$

equipped with the norm $\|u\|_{V_S}=\|u_{ss}\|_2$, and

$$W_S=w\in V_S\cap H^4(0,L)|w_{ss}(0,t)=0,\quad w_{ss}(L,t)=0 \tag{74}$$

equipped with the norm $\|u\|_{W_S}=\|w_{ss}\|_2+\|w_{ssss}\|_2$ where $\|\cdot\|_p$ denotes the L^p norms. From the Poincare' inequality, it follows that $\|\cdot\|_{V_S}$ and $\|\cdot\|_{W_S}$ are equivalent to the standard norms of $H^2(0,L)$ and $H^4(0,L)$, respectively. Next, we consider $\phi\in V_S$. Now inner producting both sides of the first equation of (24) by ϕ then integrating from 0 to L by parts result in

$$m_o\int_0^L w_{tt}\cdot\phi\,\mathrm{d}s+\int_0^L F\cdot\phi_s\,\mathrm{d}s-\int_0^L q\cdot\phi\,\mathrm{d}s-F(L,t)\cdot\phi(L,t)=0 \tag{75}$$

where $F(L,t)$ is given in (47). We will use the Galerkin approximation to show that for all $\phi\in V_S$ there exists $w\in W_S$ such that (75) holds. Let ϕ^j be a vector whose each component is a complete orthogonal system of W_S for which $\{w(s,t_0),w_t(s,t_0)\}\in\mathrm{Span}\{\phi^1,\phi^2\}$. For each $n\in N$, let $W_{Sn}=\mathrm{Span}\{\phi^1,\phi^2,\ldots,\phi^n\}$. We search for a function

$w^n(s,t)=\sum_{j=1}^{n}k^j(t)\phi^j$ such that for any $\phi\in W_{Sn}$, it satisfies the approximate closed loop system

$$m_o\int_0^L w_{tt}^n\cdot\phi\,\mathrm{d}s+\int_0^L F^n\cdot\phi_s\,\mathrm{d}s-\int_0^L q^n\cdot\phi\,\mathrm{d}s-F^n(L,t)\cdot\phi(L,t)=0,\quad s\in(0,L)$$

$$r_s^n\times(Bw_{sss}^n+F^n)=0,\quad s\in(0,L)$$

$$\dot{x}_1^n=x_2^n$$

$$\begin{aligned}\dot{x}_2^n=M_H^{-1}&\left(-B_Hx_2^n-K_1\left(w_t^n(L,t)+\frac{\gamma L}{m_o}w_s^n(L,t)-\frac{\gamma}{2m_o}w^n(L,t)\right)-\left(B_H+\frac{\gamma}{2m_o}M_H\right)\right.\\&\left.\times\left(\frac{\gamma L}{m_o}w_s^n(L,t)-\frac{\gamma}{2m_o}w^n(L,t)\right)-\frac{\gamma L}{m_o}M_Hw_{st}^n(L,t)+T_0r_s^n(L,t)-\varDelta_e^n+x_{3e}^n\right)\end{aligned}$$

$$\dot{x}_{3e}^n=-w_t^n(L,t)-\frac{\gamma L}{m_o}w_s^n(L,t)+\frac{\gamma}{2m_o}w^n(L,t)-K_2x_{3e}^n+A_H\bar{V}_H^{-1}x_{4e}^n-\left(\frac{\partial\alpha_1^n}{\partial x_2^n}-\frac{\partial\alpha_1^n}{\partial\hat{\varDelta}^n}K\right)M_H^{-1}\varDelta_e^n$$

$$\dot{x}_{4e}^n=-A_H\bar{V}_H^{-1}x_{3e}^n-K_3x_{4e}^n-\left(\frac{\partial\alpha_2^n}{\partial\hat{\varDelta}^n}K-\frac{\partial\alpha_2^n}{\partial x_2^n}\right)M_H^{-1}\varDelta_e^n$$

$$\dot{\varDelta}_e^n=-\frac{k}{m_H}\varDelta_e^n+\dot{\varDelta}^n$$

$$w^n(0,t)=0,\quad w_{ss}^n(0,t)=0,\quad w_{ss}^n(L,t)=0\tag{76}$$

where $\bullet^n$ denotes $\bullet$ with its arguments replaced by the approximate arguments. For example α_1^n denotes α_1 with its arguments $w(L,t)$, $w_t(L,t)$, $w_s(L,t)$, $w_{st}(L,t)$, $r_s(L,t)$ and $\hat{\varDelta}$ replaced by $w^n(L,t)$, $w_t^n(L,t)$, $w_s^n(L,t)$, $w_{st}^n(L,t)$, $r_s^n(L,t)$ and $\hat{\varDelta}^n$, respectively. The approximate closed loop system (76) with with the initial conditions

$$w^n(s,t_0)=w(s,t_0),\quad w_t^n(s,t_0)=w_t(s,t_0)\tag{77}$$

which are possible since each element of $(w(s,t_0),w_t(s,t_0))$ belongs to W_{Sn} for $n\geq 2$, forms in fact a system of ordinary differential equations in the variable t, which has a local solution in $[0,t_n)$. After the estimates below, the approximate solution will be extended to the interval $[0,T]$ for any given $T>0$.

Estimate I : *Upper bound of* $\int_0^L w_t^n\cdot w_t^n\,\mathrm{d}s+\int_0^L w_{ss}^n\cdot w_{ss}^n\,\mathrm{d}s$. In (76), we take $\phi=w_t^n$ and consider the following Lyapunov function candidate:

$$\begin{aligned}L_1=&\frac{m_o}{2}\int_0^L w_t^n\cdot w_t^n\,\mathrm{d}s+\frac{B}{2}\int_0^L w_{ss}^n\cdot w_{ss}^n\mathrm{d}s+\frac{\lambda}{2}\int_0^L w_s^n\cdot w_s^n\,\mathrm{d}s+\gamma\int_0^L sw_t^n\cdot w_s^n\,\mathrm{d}s-\frac{\gamma}{2}\int_0^L w_t^n\cdot w^n\,\mathrm{d}s\\&+\frac{1}{2}\left[w_t^n(L,t)+\frac{\gamma L}{m_o}w_s^n(L,t)-\frac{\gamma}{2m_o}w^n(L,t)\right]^{\mathrm{T}}M_H\left[w_t^n(L,t)+\frac{\gamma L}{m_o}w_s^n(L,t)-\frac{\gamma}{2m_o}w^n(L,t)\right]\\&+\frac{1}{2}x_{3e}^{nT}x_{3e}^n+\frac{1}{2}x_{4e}^{nT}x_{4e}^n+\frac{\chi}{2}\varDelta_e^{nT}\varDelta_e^n\end{aligned}\tag{78}$$

where λ and γ are positive constants specified as in Section 3, the positive constant χ will be specified later. Indeed, as in Section 3, the function L_1 is a proper function (i.e. positive definite and radially unbounded) as long as the constants λ and γ are taken such that they satisfy the conditions specified in (38). We use the same technique in Section 3 to calculate the time derivative of the function L_1 along the solutions of (76) as follows:

$$\begin{aligned}\dot{L}_1\leq&-c_3\left(w_t^n(L,t)+\frac{\gamma L}{m_o}w_s^n(L,t)-\frac{\gamma}{2m_o}w^n(L,t)\right)\cdot\left(w_t^n(L,t)+\frac{\gamma L}{m_o}w_s^n(L,t)-\frac{\gamma}{2m_o}w^n(L,t)\right)\\&-c_4w_t^n(L,t)\cdot w_t^n(L,t)-c_5w_s^n(L,t)\cdot w_s^n(L,t)-c_6w^n(L,t)\cdot w^n(L,t)-c_7\int_0^L w_t^n\cdot w_t^n\,\mathrm{d}s\\&-c_8\int_0^L w_s^n\cdot w_s^n\,\mathrm{d}s-c_9\int_0^L w_{ss}^n\cdot w_{ss}^n\,\mathrm{d}s-x_{3e}^{nT}K_2x_{3e}^n-x_{4e}^{nT}K_3x_{4e}^n-x_{4e}^{nT}\left(\frac{\partial\alpha_2^n}{\partial\hat{\varDelta}^n}K-\frac{\partial\alpha_2^n}{\partial x_2^n}\right)M_H^{-1}\varDelta_e^n\\&-\left(w_t^n(L,t)+\frac{\gamma L}{m_o}w_s^n(L,t)-\frac{\gamma}{2m_o}w^n(L,t)\right)\varDelta_e^n+x_{3e}^{nT}\left(\frac{\partial\alpha_1^n}{\partial x_2^n}-\frac{\partial\alpha_1^n}{\partial\hat{\varDelta}^n}K\right)M_H^{-1}\varDelta_e^n\\&+\left(\frac{\gamma\varDelta_e^n\cdot r_s^n(L,t)}{4m_o}-\frac{\gamma x_{3e}^n\cdot r_s^n(L,t)}{4m_o}\right)\int_0^L w_s^n\cdot w_s^n\,\mathrm{d}s+\varOmega_1^n-\chi\varDelta_e^{nT}KM_H^{-1}\varDelta_e^n+\chi\varDelta_e^{nT}\dot{\varDelta}^n\end{aligned}\tag{79}$$

where Ω_1^n is Ω_1 given in (41) with all of its arguments replaced by their approximations, i.e.

$$\Omega_1^n = \int_0^L \left(q^n \cdot w_t^n + \frac{\gamma q^n \cdot w_s^n s}{m_o} - \frac{\gamma q^n \cdot w^n}{2m_o} - \frac{\gamma w_s^n \cdot w_s^n}{4m_o} \int_s^L q^n(\sigma, t, w_t^n, r_\sigma(\sigma,t))^n \cdot r_\sigma^n(\sigma,t)\,\mathrm{d}\sigma \right) \mathrm{d}s \tag{80}$$

Now by substituting the expression of q given in (9) into (80), noting that $r_s^n \cdot r_s^n = 1$ for all $s \in [0,L]$ and $t \geq t_0 \geq 0$ and using Lemma 1, there exists a positive constant ϱ_1 such that

$$\Omega_1^n \leq \varrho_1 \left(\int_0^L w_t^n \cdot w_t^n \,\mathrm{d}s + \int_0^L w_s^n \cdot w_s^n \,\mathrm{d}s \right) + \frac{1}{4\varrho_1} Q_1 \tag{81}$$

where the nonnegative constant Q_1 depends on the maximum value of $\|V\|$ with V being the liquid flow, see (9). Moreover, we need to bound the rest of cross terms in (79) as follows:

$$\begin{aligned}
&\left| x_{4e}^{nT} \left(\frac{\partial \alpha_2^n}{\partial \hat{\Delta}^n} K - \frac{\partial \alpha_2^n}{\partial x_2^n} \right) M_H^{-1} \Delta_e^n \right| \leq \varrho_2 x_{4e}^{nT} x_{4e}^n + \frac{1}{4\varrho_2} \left\| \left(\frac{\partial \alpha_2^n}{\partial \hat{\Delta}^n} K - \frac{\partial \alpha_2^n}{\partial x_2^n} \right) M_H^{-1} \right\|^2 \|\Delta_e^n\|^2 \\
&\left| \left(w_t^n(L,t) + \frac{\gamma L}{m_o} w_s^n(L,t) - \frac{\gamma}{2m_o} w^n(L,t) \right) \Delta_e^n \right| \leq \varrho_2 \left\| \left(w_t^n(L,t) + \frac{\gamma L}{m_o} w_s^n(L,t) - \frac{\gamma}{2m_o} w^n(L,t) \right) \right\|^2 + \frac{1}{4\varrho_2} \|\Delta_e^n\|^2 \\
&\left| x_{3e}^{nT} \left(\frac{\partial \alpha_1^n}{\partial x_2^n} - \frac{\partial \alpha_1^n}{\partial \hat{\Delta}^n} K \right) M_H^{-1} \Delta_e^n \right| \leq \varrho_2 x_{3e}^{nT} x_{3e}^n + \frac{1}{4\varrho_2} \left\| \left(\frac{\partial \alpha_1^n}{\partial x_2^n} - \frac{\partial \alpha_1^n}{\partial \hat{\Delta}^n} K \right) M_H^{-1} \right\|^2 \|\Delta_e^n\|^2 \\
&\left| \left(\frac{\gamma \Delta_e^n \cdot r_s^n(L,t)}{4m_o} - \frac{\gamma x_{3e}^n \cdot r_s^n(L,t)}{4m_o} \right) \int_0^L w_s^n \cdot w_s^n \,\mathrm{d}s \right| \leq \varrho_3 \int_0^L w_s^n \cdot w_s^n \,\mathrm{d}s + \frac{\gamma^2 L^2}{32 m_o^2 \varrho_3} (\|\Delta_e^n\|^2 + x_{3e}^{nT} x_{3e}^n) \\
&|\Delta_e^{nT} \dot{\hat{\Delta}}^n| \leq \varrho_4 \|\Delta_e^n\|^2 + \frac{\chi^2}{4\varrho_4} \|\dot{\hat{\Delta}}^n\|^2
\end{aligned} \tag{82}$$

where Lemma 1 has been used to prove the second last inequality of (82), and $\varrho_i,\ i = 1, \ldots, 4$ are positive constants to be specified later. On the other hand, from (42) and (55), it is seen that α_1^n and α_2^n are of at most linear in x_2^n and $\hat{\Delta}^n$. This implies that there exist constants M_1 and M_2 such that

$$\left\| \left(\frac{\partial \alpha_2^n}{\partial \hat{\Delta}^n} K - \frac{\partial \alpha_2^n}{\partial x_2^n} \right) M_H^{-1} \right\|^2 \leq M_1, \quad \left\| \left(\frac{\partial \alpha_1^n}{\partial x_2^n} - \frac{\partial \alpha_1^n}{\partial \hat{\Delta}^n} K \right) M_H^{-1} \right\|^2 \leq M_2, \quad \forall s \in [0,L], \ \ t \geq t_0 \geq 0 \tag{83}$$

Now substituting (83), (82) and (80) into (79) results in

$$\begin{aligned}
\dot{L}_1 \leq & -(c_3 - \varrho_2) \left\| \left(w_t^n(L,t) + \frac{\gamma L}{m_o} w_s^n(L,t) - \frac{\gamma}{2m_o} w^n(L,t) \right) \right\|^2 - c_4 w_t^n(L,t).w_t^n(L,t) \\
& - c_5 w_s^n(L,t).w_s^n(L,t) - c_6 w^n(L,t).w^n(L,t) - (c_7 - \varrho_1) \int_0^L w_t^n.w_t^n \,\mathrm{d}s - (c_8 - \varrho_1 - \varrho_3) \\
& \times \int_0^L w_s^n \cdot w_s^n \,\mathrm{d}s - c_9 \int_0^L w_{ss}^n \cdot w_{ss}^n \,\mathrm{d}s - \left(\lambda_{\min}(K_2) - \varrho_2 - \frac{\gamma^2 L^2}{32 m_o^2 \varrho_3} \right) x_{3e}^n \cdot x_{3e}^n - (\lambda_{\min}(K_3) - \varrho_2) \\
& \times x_{4e}^n.x_{4e}^n - \left(\chi \lambda_{\min}(K M_H^{-1}) - \frac{M_1 + M_2 + 1}{4\varrho_2} - \frac{\gamma^2 L^2}{32 m_o^2 \varrho_3} - \varrho_4 \right) \Delta_e^n.\Delta_e^n + \frac{\chi^2}{4\varrho_4} \|\dot{\hat{\Delta}}^n\|^2 + \frac{1}{4\varrho_1} Q_1
\end{aligned} \tag{84}$$

where $\lambda_{\min}(\bullet)$ denotes the minimum eigenvalue of $\bullet$. We pick $\varrho_i,\ i = 1, \ldots, 4$ and χ such that all the constants $(c_3 - \varrho_2), (c_7 - \varrho_1), (c_8 - \varrho_1 - \varrho_3), c_9, (\lambda_{\min}(K_2) - \varrho_2 - \gamma^2 L^2 / 32 m_o^2 \varrho_3), \lambda_{\min}(K_3) - \varrho_2, (\chi \lambda_{\min}(K M_H^{-1}) - (M_1 + M_2 + 1)/4\varrho_2 - \gamma^2 L^2 / 32 m_o^2 \varrho_3 - \varrho_4)$ are strictly positive. Now from definition of L_1, see (78) and (84), we have

$$\dot{L}_1 \leq -\frac{\bar{c}_1}{\bar{c}_2} L_1 + \bar{c}_3 \Rightarrow L_1(t) \leq \left(L_1(t_0) + \frac{\bar{c}_2 \bar{c}_3}{\bar{c}_1} \right) \mathrm{e}^{-\bar{c}_1/\bar{c}_2} + \frac{\bar{c}_1 \bar{c}_3}{\bar{c}_2}, \quad \forall t \geq t_0 \geq 0 \tag{85}$$

where

$$\begin{aligned}
\bar{c}_1 = \min & \left((c_3 - \varrho_2), (c_7 - \varrho_1), (c_8 - \varrho_1 - \varrho_3), c_9, \left(\lambda_{\min}(K_2) - \varrho_2 - \frac{\gamma^2 L^2}{32 m_o^2 \varrho_3} \right), (\lambda_{\min}(K_3) - \varrho_2), \right. \\
& \left. \left(\chi \lambda_{\min}(K M_H^{-1}) - \frac{M_1 + M_2 + 1}{4\varrho_2} - \frac{\gamma^2 L^2}{32 m_o^2 \varrho_3} - \varrho_4 \right) \right)
\end{aligned}$$

$$\bar{c}_2 = \tfrac{1}{2} \max((m_o - \gamma(L+1)), (\lambda - \gamma(L + L^2)), B, 1, \chi)$$

$$\bar{c}_3 = \max\left(\frac{\chi^2}{4\varrho_4}\|\dot{\Delta}^n\|^2 + \frac{1}{4\varrho_1}Q_1\right) \tag{86}$$

Hence from (85) and (78), we deduce that there exists a nonnegative constant P_1 such that

$$\int_0^L w_t^n \cdot w_t^n \,\mathrm{d}s + \int_0^L w_s^n \cdot w_s^n \,\mathrm{d}s + \int_0^L w_{ss}^n \cdot w_{ss}^n \,\mathrm{d}s \le P_1, \quad \forall t \in [0,T], \;\; n \in N \tag{87}$$

Estimate II : Upper bound of $w_{tt}(s,t_0)$ in L^2-norm. In the first equation of (76), taking $\phi = w_{tt}^n(s,t_0)$ and $t = t_0$, and integrating by parts give

$$m_o \int_0^L w_{tt}^n(s,t_0) \cdot w_{tt}^n(s,t_0)\,\mathrm{d}s - \int_0^L F_s^n(s,t_0) \cdot w_{tt}^n(s,t_0)\,\mathrm{d}s - \int_0^L q^n(s,t_0) \cdot w_{tt}^n(s,t_0)\,\mathrm{d}s = 0 \tag{88}$$

for all $s \in (0,L)$. Let us calculate the term $\int_0^L F_s^n(s,t_0) \cdot w_{tt}^n(s,t_0)\,\mathrm{d}s$ in (88). Using Lemma 1 gives

$$\begin{aligned} F_s^n(s,t_0) \cdot w_{tt}^n(s,t_0) = &-Bw_{ssss}^n(s,t_0) \cdot r_s^n(s,t_0) + Bw_{sss}^n(s,t_0) \cdot r_s^n(s,t_0) w_{tt}^n(s,t_0) \cdot w_{ss}^n(s,t_0) \\ &+ F^n(s,t)_0 \cdot r_s^n(s,t_0) w_{tt}^n(s,t_0) \cdot w_{ss}^n(s,t_0), \quad \forall (s,t_0) \in ([0,L],\mathbb{R}^+) \end{aligned}$$

$$\begin{aligned} F^n(s,t_0) \cdot r_s^n(s,t_0) = &F^n(L,t_0) \cdot r_s^n(L,t_0) - \frac{B}{2} w_{ss}^n(s,t_0) \cdot w_{ss}^n(s,t_0) \\ &+ \int_s^L q^n(\sigma,t_0,w_t^n(\sigma,t_0),r_\sigma^n(\sigma,t_0)) \cdot r_\sigma^n(\sigma,t_0)\,\mathrm{d}\sigma, \quad \forall (s,t_0) \in ((0,L),\mathbb{R}^+) \end{aligned} \tag{89}$$

and using (48) gives

$$\begin{aligned} F^n(L,t_0).r_s^n(L,t_0) = &-r_s^{nT}(L,t_0)\left(K_1 + B_H + \frac{\gamma}{2m_o}M_H\right)\left(\frac{\gamma L}{m_o} w_s^n(L,t_0) - \frac{\gamma}{2m_o} w^n(L,t_0)\right) \\ &- \Delta_e^n.r_s^n(L,t_0) + T_0 + x_{3e}^n(t_0).r_s^n(L,t_0) \end{aligned} \tag{90}$$

Now by substituting (90) into the second equation of (91) then to the first equation of (91) then to (88) and using completion of squares and the estimate I, see (85), it is readily shown that there exists a nonnegative constant P_2 such that

$$\int_0^L w_{tt}^n(s,t_0).w_{tt}^n(s,t_0)\,\mathrm{d}s \le P_2, \quad \forall t \in [0,T], \;\; n \in N \tag{91}$$

Estimate III : Upper bound of $w_{tt}(s,t)$ and $w_{sst}(s,t)$ in L^2-norm. To estimate the upper bound of these terms, we use difference approach. Let us fix t and ξ such that $\xi < T - t$. Now taking the difference of (76) with $t = t + \xi$ and $t = t$, and then letting $\phi = w_t^n(s,t+\xi) - w_t^n(s,t)$ result in

$$\begin{aligned} &\frac{m_o}{2}\int_0^L \frac{\mathrm{d}}{\mathrm{d}t}[(w_t^n(s,t+\xi) - w_t^n(s,t)) \cdot (w_t^n(s,t+\xi) - w_t^n(s,t))]\,\mathrm{d}s \\ &+ \int_0^L (F^n(s,t+\xi) - F^n(s,t)) \cdot (w_{st}^n(s,t+\xi) - w_{st}^n(s,t))\,\mathrm{d}s \\ &- (F^n(L,t+\xi) - F^n(L,t)) \cdot (w_t^n(L,t+\xi) - w_t^n(L,t)) \\ &+ \int_0^L (q(s,t+\xi) - q(s,t)) \cdot (w_t^n(s,t+\xi) - w_t^n(s,t))\,\mathrm{d}s = 0 \end{aligned} \tag{92}$$

To deal with the second integration term in (92), we proceed the second equation of (76) as follows:

$$\begin{aligned} r_s^n(s,t) \times (Bw_{sss}^n(s,t) + F^n(s,t)) = 0 &\Rightarrow w_{st}(s,t) \times (r_s^n(s,t) \times (Bw_{sss}^n(s,t) + F^n(s,t))g) = 0 \\ &\Rightarrow F^n(s,t) \cdot w_{st}(s,t) = -Bw_{sss}^n \cdot w_{st}(s,t) \end{aligned} \tag{93}$$

for all $(s,t) \in ((0,L),\mathbb{R}^+)$ since $w_{st}^n(s,t) \cdot r_s^n(s,t) = 0$ for all $(s,t) \in ((0,L),\mathbb{R}^+)$. Moreover, using (47) results in

$$\begin{aligned} F^n(L,t) \cdot w_t^n(L,t) = &-M_H w_{tt}^n(L,t) \cdot w_t^n(L,t) - B_H w_t^n(L,t) \cdot w_t^n(L,t) - K_1\left(w_t^n(L,t) + \frac{\gamma L}{m_o} w_s^n(L,t)\right. \\ &\left.- \frac{\gamma}{2m_o} w^n(L,t)\right) \cdot w_t^n(L,t) - \left(B_H + \frac{\gamma}{2m_o}M_H\right)\left(\frac{\gamma L}{m_o} w_s^n(L,t) - \frac{\gamma}{2m_o} w^n(L,t)\right) \cdot w_t^n(L,t) \\ &- \frac{\gamma L}{m_o} M_H w_{st}(L,t) \cdot w_t^n(L,t) \cdot w_t^n(L,t) - \Delta_e^n + T_0 r_s^n(L,t) + x_{3e}^n \cdot w_t^n(L,t) \end{aligned} \tag{94}$$

Since the initial values $w(s,t_0)$ and $w_t(s,t_0)$ are sufficiently smooth, $w(0,t) + r^0 = 0$, $w_{ss}(0,t) = 0$, $w_{ss}(L,t) = 0$ for $w \in W_S$ and all the terms $\int_0^L w_t^n(s,t) \cdot w_t^n(s,t)\,\mathrm{d}s$, $\int_0^L w_s^n(s,t) \cdot w_s^n(s,t)\,\mathrm{d}s$, and $\int_0^L w_{ss}^n(s,t) \cdot w_{ss}^n(s,t)\,\mathrm{d}s$ are bounded, see Estimate I, using the Mean Value Theorem and Lemmas 2 and 3 shows readily that there exists a nonnegative constant M_3 such that

$$\frac{\mathrm{d}\Phi^n}{\mathrm{d}t}(t,\xi) \le M_3 \Phi^n(t,\xi) \Rightarrow \Phi(t,\xi) \le \Phi(t_0,\xi)\mathrm{e}^{M_3(t-t_0)} \tag{95}$$

where

$$\Phi^n(t,\xi)=\frac{m_o}{2}\int_0^L (w_t^n(s,t+\xi)-w_t^n(s,t))\cdot(w_t^n(s,t+\xi)-w_t^n(s,t))\,\mathrm{d}s \\ +\frac{B}{2}\int_0^L (w_{ss}^n(s,t+\xi)-w_{ss}^n(s,t))\cdot(w_{ss}^n(s,t+\xi)-w_{ss}^n(s,t))\,\mathrm{d}s \tag{96}$$

Dividing both sides of the last inequality in (95) by ξ^2 then taking the limit $\xi\to 0$ gives

$$m_o\int_0^L w_{tt}^n(s,t)\cdot w_{tt}^n(s,t)\,\mathrm{d}s+B\int_0^L w_{sst}^n(s,t)\cdot w_{sst}^n(s,t)\,\mathrm{d}s \\ \le\left[m_o\int_0^L w_{tt}^n(s,t_0)\cdot w_{tt}^n(s,t_0)\,\mathrm{d}s+B\int_0^L w_{sst}^n(s,t_0)\cdot w_{sst}^n(s,t_0)\,\mathrm{d}s\right]\mathrm{e}^{M_3(t-t_0)} \tag{97}$$

for all $t_0\le t\le T$. Now from the estimates given in (87) and (91), we can deduce from (95) that there exists $P_3>0$ depending on T such that

$$m_o\int_0^L w_{tt}^n(s,t)\cdot w_{tt}^n(s,t)\,\mathrm{d}s+B\int_0^L w_{sst}^n(s,t)\cdot w_{sst}^n(s,t)\,\mathrm{d}s\le P_3 \tag{98}$$

From the estimates given in (87), (91) and (95), we can use the Lions–Aubin theorem to get the necessary compactness to pass the nonlinear system (76) to the limit. Then it is a matter of routine to conclude the existence of global solutions in $[0,T]$.

Uniqueness. Let u and v be two solutions of the closed loop system (64). Letting $z=u-v$, we have $z(s,t_0)=0$ and $z_t(s,t_0)=0$ and from (75) we have

$$m_o\int_0^L z_{tt}\cdot\phi\,\mathrm{d}s+\int_0^L (F|_{w=u}-F|_{w=v})\cdot\phi_s\,\mathrm{d}s-\int_0^L (q|_{w=u}-|_{w=v})\cdot\phi\,\mathrm{d}s \\ -(F(L,t)|_{w=u}-F(L,t)|_{w=v})\cdot\phi(L,t)=0 \tag{99}$$

where the expression of $F(s,t)$ is given in (47). By taking $\phi=z_t(s,t)$ in (99) and using the Mean Value Theorem and passing of the limit of all the estimates given in in (87), (91) and (95) previously, we readily have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^L z_t\cdot z_t\,\mathrm{d}s+\int_0^L z_{ss}\cdot z_{ss}\,\mathrm{d}s\right)\le M_4\left(\int_0^L z_t\cdot z_t\,\mathrm{d}s+\int_0^L z_{ss}\cdot z_{ss}\,\mathrm{d}s\right) \tag{100}$$

where M_4 is a positive constant. Since $z(s,t_0)=0$ and $z_t(s,t_0)=0$, using Gronwall's Lemma shows that $z=0$, i.e. $u=v$ for all $t\ge t_0\ge 0$ and $s\in[0,L]$.

C.2. Proof of convergence

We consider the following Lyapunov function candidate:

$$W=W_3+\frac{v}{2}\Delta_e^{\mathrm{T}}\Delta_e \tag{101}$$

where W_3 is given in (59), and v is a positive constant to be chosen later. Differentiating both sides of (101) along the solutions of (63) and (45) gives

$$\begin{aligned}\dot W\le&-c_3\left(w_t(L,t)+\frac{\gamma L}{m_o}w_s(L,t)-\frac{\gamma}{2m_o}w(L,t)\right)\cdot\left(w_t(L,t)+\frac{\gamma L}{m_o}w_s(L,t)-\frac{\gamma}{2m_o}w(L,t)\right)\\&-c_4w_t(L,t)\cdot w_t(L,t)-c_5w_s(L,t)\cdot w_s(L,t)-c_6w(L,t)\cdot w(L,t)-c_7\int_0^L w_t\cdot w_t\,\mathrm{d}s\\&-c_8\int_0^L w_s\cdot w_s\,\mathrm{d}s-c_9\int_0^L w\cdot.w_{ss}\,\mathrm{d}s-x_{3e}^{\mathrm{T}}K_2x_{3e}-x_{4e}^{\mathrm{T}}K_3x_{4e}-x_{4e}^{\mathrm{T}}\left(\frac{\partial\alpha_2}{\partial\hat\Delta}K-\frac{\partial\alpha_2}{\partial x_2^n}\right)M_H^{-1}\Delta_e\\&-\left(w_t(L,t)+\frac{\gamma L}{m_o}w_s(L,t)-\frac{\gamma}{2m_o}w(L,t)\right)\Delta_e+x_{3e}^{\mathrm{T}}\left(\frac{\partial\alpha_1}{\partial x_2}-\frac{\partial\alpha_1}{\partial\hat\Delta}K\right)M_H^{-1}\Delta_e\\&+\Omega_1\left(\frac{\gamma\Delta_e\cdot r_s(L,t)}{4m_o}-\frac{\gamma x_{3e}.r_s(L,t)}{4m_o}\right)\int_0^L w_s\cdot w_s\,\mathrm{d}s-v\Delta_e^{\mathrm{T}}KM_H^{-1}\Delta_e+v\Delta_e^{\mathrm{T}}\dot\Delta\end{aligned} \tag{102}$$

Now by substituting the expression of q given in (9) into (41), noting that $r_s\cdot r_s=1$ for all $s\in[0,L]$ and $t\ge t_0\ge 0$ and using Lemma 1, there exists a positive constant ρ_1 such that

$$\Omega_1\le\rho_1\left(\int_0^L w_t\cdot w_t\,\mathrm{d}s+\int_0^L w_s\cdot w_s\,\mathrm{d}s\right)+\frac{1}{4\rho_1}G_1 \tag{103}$$

where the nonnegative constant G_1 depends on the maximum value of $\|V\|$ with V being the liquid flow, see (9). Moreover, we need to bound the rest crossed terms in (102) as follows:

$$\left|x_{4e}^{\mathrm{T}}\left(\frac{\partial\alpha_2}{\partial\hat{\Delta}}K-\frac{\partial\alpha_2}{\partial x_2}\right)M_H^{-1}\Delta_e\right|\le\rho_2x_{4e}^{\mathrm{T}}x_{4e}+\frac{1}{4\rho_2}\left\|\left(\frac{\partial\alpha_2}{\partial\hat{\Delta}}K-\frac{\partial\alpha_2}{\partial x_2}\right)M_H^{-1}\right\|^2\|\Delta_e\|^2$$

$$\left|\left(w_t(L,t)+\frac{\gamma L}{m_o}w_s(L,t)-\frac{\gamma}{2m_o}w(L,t)\right)\Delta_e\right|\le\rho_2\left\|\left(w_t(L,t)+\frac{\gamma L}{m_o}w_s(L,t)-\frac{\gamma}{2m_o}w(L,t)\right)\right\|^2+\frac{1}{4\rho_2}\|\Delta_e\|^2$$

$$\left|x_{3e}^{\mathrm{T}}\left(\frac{\partial\alpha_1}{\partial x_2}-\frac{\partial\alpha_1}{\partial\hat{\Delta}}K\right)M_H^{-1}\Delta_e\right|\le\rho_2x_{3e}^{\mathrm{T}}x_{3e}+\frac{1}{4\rho_2}\left\|\left(\frac{\partial\alpha_1}{\partial x_2}-\frac{\partial\alpha_1}{\partial\hat{\Delta}}K\right)M_H^{-1}\right\|^2\|\Delta_e\|^2$$

$$\left|\left(\frac{\gamma\Delta_e\cdot r_s(L,t)}{4m_o}-\frac{\gamma x_{3e}\cdot r_s(L,t)}{4m_o}\right)\int_0^L w_s\cdot w_s\,\mathrm{d}s\right|\le\rho_3\int_0^L w_s\cdot w_s\,\mathrm{d}s+\frac{\gamma^2L^2}{32m_o^2\rho_3}(\|\Delta_e\|^2+x_{3e}^{\mathrm{T}}x_{3e})$$

$$\left|\Delta_e^{\mathrm{T}}\dot{\Delta}\right|\le\rho_4\|\Delta_e\|^2+\frac{v^2}{4\rho_4}\|\dot{\Delta}\|^2 \tag{104}$$

where Lemma 1 has been used to prove the second last inequality of (104), and $\rho_i,\ i=1,\ldots,4$ are positive constants to be specified later. On the other hand, from (42) and (55), it is seen that α_1 and α_2 are of at most linear in x_2 and $\hat{\Delta}$. This implies that there exist constants N_1 and N_2 such that

$$\left\|\left(\frac{\partial\alpha_2}{\partial\hat{\Delta}}K-\frac{\partial\alpha_2}{\partial x_2}\right)M_H^{-1}\right\|^2\le N_1,\quad\left\|\left(\frac{\partial\alpha_1}{\partial x_2}-\frac{\partial\alpha_1}{\partial\hat{\Delta}}K\right)M_H^{-1}\right\|^2\le N_2,\quad\forall s\in[0,L],\ \ t\ge t_0\ge0 \tag{105}$$

Now substituting (105), (104) and (103) into (102) results in

$$\begin{aligned}\dot{W}\le&-(c_3-\varrho_2)\left\|\left(w_t(L,t)+\frac{\gamma L}{m_o}w_s(L,t)-\frac{\gamma}{2m_o}w(L,t)\right)\right\|^2-c_4w_t(L,t)\cdot w_t(L,t)\\&-c_5w_s(L,t)\cdot w_s(L,t)-c_6w(L,t)\cdot w^n(L,t)-(c_7-\rho_1)\int_0^L w_t\cdot w_t\,\mathrm{d}s-(c_8-\rho_1-\rho_3)\\&\times\int_0^L w_s\cdot w_s\,\mathrm{d}s-c_9\int_0^L w_{ss}\cdot w_{ss}\,\mathrm{d}s-\left(\lambda_{\min}(K_2)-\rho_2-\frac{\gamma^2L^2}{32m_o^2\rho_3}\right)x_{3e}.x_{3e}-(\lambda_{\min}(K_3)-\varrho_2)\\&\times x_{4e}.x_{4e}-\left(v\lambda_{\min}(KM_H^{-1})-\frac{N_1+N_2+1}{4\rho_2}-\frac{\gamma^2L^2}{32m_o^2\rho_3}-\rho_4\right)\Delta_e.\Delta_e+\frac{v^2}{4\rho_4}\|\dot{\Delta}\|^2+\frac{1}{4\rho_1}G_1\end{aligned} \tag{106}$$

where $\lambda_{\min}(\bullet)$ denotes the minimum eigenvalue of $\bullet$. We pick $\rho_i,\ i=1,\ldots,4$ and v such that all the constants $(c_3-\rho_2)$, $(c_7-\rho_1)$, $(c_8-\rho_1-\rho_3)$, c_9, $(\lambda_{\min}(K_2)-\rho_2-\gamma^2L^2/32m_o^2\rho_3)$, $\lambda_{\min}(K_3)-\rho_2$, $(v\lambda_{\min}(KM_H^{-1})-(N_1+N_2+1)/4\rho_2-\gamma^2L^2/32m_o^2\rho_3-\rho_4)$ are strictly positive. Now from definition of W, see (101) and (106), we have

$$\dot{W}\le-\frac{\bar{c}_1}{\bar{c}_2}W+\bar{c}_3\ \Rightarrow W(t)\le\left(W(t_0)+\frac{\bar{c}_2\bar{c}_3}{\bar{c}_1}\right)\mathrm{e}^{-\bar{c}_1/\bar{c}_2}+\frac{\bar{c}_1\bar{c}_3}{\bar{c}_2},\quad\forall t\ge t_0\ge0 \tag{107}$$

where

$$\begin{aligned}\bar{c}_1=&\min\left((c_3-\rho_2),(c_7-\rho_1),(c_8-\rho_1-\rho_3),c_9,\left(\lambda_{\min}(K_2)-\rho_2-\frac{\gamma^2L^2}{32m_o^2\rho_3}\right),(\lambda_{\min}(K_3)-\rho_2),\right.\\&\left.\left(v\lambda_{\min}(KM_H^{-1})-\frac{N_1+N_2+1}{4\rho_2}-\frac{\gamma^2L^2}{32m_o^2\rho_3}-\rho_4\right)\right)\\\bar{c}_2=&\tfrac{1}{2}\max((m_o-\gamma(L+1)),(\lambda-\gamma(L+L^2)),B,1,v)\end{aligned}$$

$$\bar{c}_3=\max\left(\frac{v^2}{4\rho_4}\|\dot{\Delta}\|^2+\frac{1}{4\rho_1}G_1\right) \tag{108}$$

The bound on W given in (107) combined with the definition of W, see (101), shows that

$$\int_0^L w_t(s,t)\cdot w_t(s,t)\,\mathrm{d}s\le\frac{2}{c_1}\left(W(t_0)+\frac{\bar{c}_2\bar{c}_3}{\bar{c}_1}\right)\mathrm{e}^{-\bar{c}_1/\bar{c}_2}+\frac{2\bar{c}_1\bar{c}_3}{c_1\bar{c}_2}$$

$$\int_0^L w_s(s,t)\cdot w_s(s,t)\,\mathrm{d}s\le\frac{2}{c_2}\left(W(t_0)+\frac{\bar{c}_2\bar{c}_3}{\bar{c}_1}\right)\mathrm{e}^{-\bar{c}_1/\bar{c}_2}+\frac{2\bar{c}_1\bar{c}_3}{c_2\bar{c}_2}$$

$$\int_0^L w_{ss}(s,t) \cdot w_{ss}(s,t)\,\mathrm{d}s \le \frac{2}{B}\left(W(t_0) + \frac{\bar{c}_2\bar{c}_3}{\bar{c}_1}\right)\mathrm{e}^{-\bar{c}_1/\bar{c}_2} + \frac{2\bar{c}_1\bar{c}_3}{B\bar{c}_2} \tag{109}$$

Since the initial values of $w_t(s,t_0)$, $w_s(s,t_0)$, $w_{ss}(s,t_0)$ for all $s \in [0,L]$ are bounded and sufficiently smooth, and $x_3(t_0)$ and $x_4(t_0)$ are bounded as well, the right hand sides of all inequalities in (109) are bounded. Hence, the right hand sides of the first, second and third inequalities in (109) are bounded and exponentially converge to $2\bar{c}_1\bar{c}_3/c_2\bar{c}_2$, $2\bar{c}_1\bar{c}_3/c_2\bar{c}_2$ and $2\bar{c}_1\bar{c}_3/B\bar{c}_2$, respectively. This implies that the left hand sides of the first, second and third inequalities in (109) must be bounded and must exponentially converge to $2\bar{c}_1\bar{c}_3/c_2\bar{c}_2$, $2\bar{c}_1\bar{c}_3/c_2\bar{c}_2$and $2\bar{c}_1\bar{c}_3/B\bar{c}_2$, respectively. Next, we use Lemmas 2 and 3 to show that $\int_0^L w(s,t) \cdot w(s.t)\,\mathrm{d}s$ and $\|w(s,t)\|$ are bounded and exponentially converge to some constant. An application of Lemma 2 gives

$$\int_0^L w(s,t) \cdot w(s \cdot t)\,\mathrm{d}s \le 2w(0,t) \cdot w(0,t) + 4L^2 \int_0^L w_s(s,t) \cdot w_s(s,t)\,\mathrm{d}s \tag{110}$$

Since $w(0,t) = 0$ and we have already proved that $\int_0^L w_s(s,t) \cdot w_s(s,t)\,\mathrm{d}s$ is bounded and exponentially converges to $2\bar{c}_1\bar{c}_3/c_2\bar{c}_2$, (110) implies that $\int_0^L w(s,t) \cdot w(s,t)\,\mathrm{d}s$ must be bounded and exponentially converges to $8L^2\bar{c}_1\bar{c}_3/c_2\bar{c}_2$. On the other hand, an application of Lemma 3 shows that

$$\max_{s\in[0,L]}(w(s,t) \cdot w(s,t)) \le w(0,t) \cdot w(0,t) + 2\sqrt{\int_0^L w(s,t) \cdot w(s,t)\,\mathrm{d}s}\sqrt{\int_0^L w_s(s,t) \cdot w_s(s,t)\,\mathrm{d}s} \tag{111}$$

Since $w(0,t) = 0$ and we have already proved that $\int_0^L w_s(s,t) \cdot w_s(s,t)\,\mathrm{d}s$ and $\int_0^L w(s,t) \cdot w(s,t)\,\mathrm{d}s$ are bounded and exponentially converge to $2\bar{c}_1\bar{c}_3/c_2\bar{c}_2$ and $8L^2\bar{c}_1\bar{c}_3/c_2\bar{c}_2$, respectively, (111) implies that $\|w(s,t)\|$ must be bounded and exponentially converges to $8L\bar{c}_1\bar{c}_3/c_2\bar{c}_2$.

For the case where there are no distributed disturbances and the disturbance Δ is constant, i.e. $q = 0$ and $\dot{\Delta} = 0$, it is directly seen from the above proof that $\int_0^L w_t(s,t) \cdot w_t(s,t)\,\mathrm{d}s$, $\int_0^L w_{ss}(s,t) \cdot w_{ss}(s,t)\,\mathrm{d}s$ and $\int_0^L w_s(s,t) \cdot w_s(s,t)\,\mathrm{d}s$ are bounded and exponentially converge to zero since $q = 0$ and $\dot{\Delta} = 0$ imply that $\bar{c}_3 = 0$, see (108). Therefore, using the same arguments as above, we have $\int_0^L w(s,t) \cdot w(s,t)\,\mathrm{d}s$ and $\|w(s,t)\|$ are bounded and exponentially converge to zero.

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